ϑ — $\langle N(t)\rangle \sim t^{\vartheta}$ (1 initially active site)

$\rho(t) \sim (p-p_c)^{\beta}$ — β critical DP exponents

$\uparrow$

$\frac{\langle N(t)\rangle}{L}$ $P(t) \sim (p-p_c)^{\beta'}$ — β'

correlation time $\xi_{\parallel} \sim |p-p_c|^{-\nu_{\parallel}}$ $z = \frac{\nu_{\parallel}}{\nu_{\perp}}$

correlation length $\xi_{\perp} \sim |p-p_c|^{-\nu_{\perp}}$ dynamical exp.

<u>MEAN FIELD APPROACH</u> $\to$ phenomenological Langevin equation for $\rho(\vec{x},t)$ density of active sites at $\vec{x}$, at time t

space-time coarse-grained average

<u>$\rho(\vec{x},t)$ uniform</u>

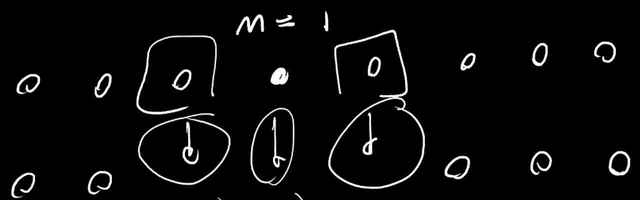
$\downarrow$

"Contact process"

↳ simplest example to model "epidemic spreading"

↳ same universality as DP

active = infected ; empty = healthy



t + "patient zero"

$t+1$ $(d = 1)$

recovery $W(1 \to 0, n) = r$

r: recovery rate

n = # of infected neighbours

infection $W(0 \to 1, n) = \lambda \frac{n}{2d}$

λ = infection rate $\quad n/2d$ = fraction of infected neighbors

$d=1$ numerical simulations: $\boxed{(\lambda/\tau)_c \simeq 3.29785}$

reinfection is allowed

MEAN FIELD with uniform $\rho(t)$

$$\frac{\partial \rho}{\partial t} = \overbrace{+\lambda \rho (1-\rho)}^{\text{gain term}} - \overbrace{\tau \rho}^{\text{loss term}} = a\rho - \lambda \rho^2$$

ρ fraction of active sites; $(1-\rho)$ if healthy; ρ infected; $a = \lambda - \tau$

$$\frac{\partial \rho}{\partial t} = \rho(a - \lambda\rho) \qquad \begin{cases} \rho_1^* = 0 \\ \rho_2^* = a/\lambda = \dfrac{\lambda - \tau}{\lambda} = 1 - \tau/\lambda \end{cases}$$

stationary solutions $\left(\left.\frac{\partial \rho}{\partial t}\right|_{\rho=\rho^*} = 0\right)$

stability $\quad \frac{\partial^2 \rho}{\partial t} = a - 2\lambda\rho \; ; \quad \left.\frac{\partial^2 \rho}{\partial t^2}\right|_{\rho=\rho_1^*} = a$

$a > 0 \to \rho_2^*$ stable $\quad \left.\frac{\partial^2 \rho}{\partial t^2}\right|_{\rho=\rho_2^*} = -a$
(ρ_1^* unstable)

$a < 0 \to \rho_1^*$ stable (ρ_2^* unstable)

$\rho_1^* = 0 \to$ inactive phase (all sites healthy)

$\rho_2^* = a/\lambda \to$ active phase (epidemic spreading)

the Mean Field critical point $a_c = 0 \to (\lambda/\tau)_c = 1$

Mean field with $\rho(\vec{x}, t)$ non uniform

(stochastic fluctuations neglected)

Langevin-like equation

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} = a\,\rho(\vec{x}, t) - \lambda \rho^2(\vec{x}, t) + \underbrace{D\,\nabla^2 \rho(\vec{x}, t)}_{\text{diffusion}} + \eta(\vec{x}, t)$$

$\langle \eta(\vec{x}, t) \rangle = 0$

$\rightarrow \langle \eta(\vec{x}, t)\, \eta(\vec{x}', t') \rangle = \Gamma\, \rho(\vec{x}, t)\, \delta(\vec{x} - \vec{x}')\, \delta(t - t')$

multiplicative noise

noise only in the active phase

inverse time

$\rightarrow \eta(\vec{x}, t) \sim \sqrt{\rho(\vec{x}, t)}$

Scaling invariance close to the critical point $a_c = 0$

scaling transformation with $\Lambda = \Delta = a$

distance from the critical point

$$\Lambda^{\beta + \nu_{\parallel}}\, \partial\rho/\partial t = \Lambda^{\beta + 1} \rho - \lambda \Lambda^{2\beta} \rho^2 + D\, \Lambda^{\beta + 2\nu_{\perp}} \nabla^2 \rho + \Lambda^{\gamma} \eta(\vec{x}, t)$$

rescaled Langevin eq.

$a = \Lambda$

$$\gamma = \frac{\beta + d\nu_{\perp} + \nu_{\parallel}}{2}$$

For the deterministic part to be "scale invariant"

$\rightarrow \partial\rho/\partial t = \Lambda^{1 - \nu_{\parallel}} \rho - \lambda \Lambda^{\beta - \nu_{\parallel}} \rho^2 + D\, \Lambda^{2\nu_{\perp} - \nu_{\parallel}} \nabla^2 \rho +$

$\boxed{\nu_\parallel = 1}$ $\quad \beta = \nu_\parallel \rightarrow \boxed{\beta = 1}$ $\quad + \Lambda^{\gamma - \beta - \nu_\parallel} \cdot \eta$

$2\nu_\perp = \nu_\parallel \rightarrow \boxed{\nu_\perp = 1/2} \rightarrow \boxed{z = \frac{\nu_\parallel}{\nu_\perp} = 2}$

mean field exponents are exact when the stochastic ~~term~~ term is irrelevant $\rightarrow \gamma - \beta - \nu_\parallel > 0$

$\left(\Lambda^{\gamma - \beta - \nu_\parallel} \rightarrow 0 \right)$

$\Lambda \rightarrow 0$

$d\nu_\perp - \beta - \nu_\parallel > 0$

$\rightarrow d > \frac{\beta + \nu_\parallel}{\nu_\perp} = 4 \rightarrow d_c = 4$

Ginzburg criterion

$(\beta' = \beta)$

upper critical dimension

$\vartheta = \frac{d\nu_\perp - 2\beta}{\nu_H} = 0$

holds only for $d \leq 4$

$d = 4$

mean field exponents with logarithmic corrections

DIFFERENT UNIVERSALITY CLASS (from DP)

(non-equil. phase transitions with absorbing phases)

- more absorbing phases

$\rightarrow$ different universality class

DP $\to$ 1 absorbing phase: all sites empty

Contact process with reinfection: all individuals healthy
(always with the same probability $\lambda \cdot n/2d$)

immunization $\to$ Reinfection probability $p_2 < p_1$
(1st infection probability p_1)

"partial" immunization $p_2 > 0$

complete immunization $\underline{\underline{p_2 = 0}}$ $\to$ SIR model

3 states
S usceptible
I nfectible
R ecovered

Infinite number of possible absorbing states
$\to$ any combination of S, R individuals

$p_2 = 0$ and $p_2 > 0$ but small enough
$\to$ different universality class DyP
(dynamical percolation)

$p_1 > p_{1,c}$ $\to$ active phase $\to$ propagation of a percolation front

$p_1 < p_{1,c}$ $\to$ inactive phase

SIR model (mean field with uniform ρ)

N = # of individuals (N constant)

$S(t) + R(t) + I(t) = N$

2 parameters (λ, r)

$$\lambda = \frac{\text{\# infected people by 1 infected individual}}{\text{unit time}} \quad \hookrightarrow S \to I$$

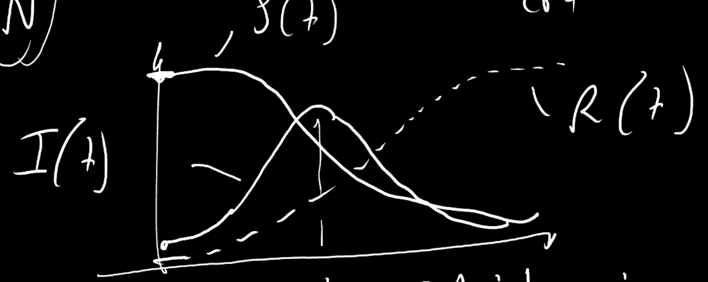
$$r = \frac{\text{recovery prob.}}{\text{unit time}} = 1/t_r \quad \text{— typical recovery time}$$

$\hookrightarrow I \to R$

fraction of susceptible individual in the population

$$\begin{cases} \dfrac{dS}{dt} = -\lambda \dfrac{S}{N} I \,; \\[2ex] \dfrac{dI}{dt} = \lambda \dfrac{S}{N} I - r I \end{cases} \qquad \frac{dR}{dt} = \gamma I$$

$$\underbrace{N = S + R + I}$$



At epidemic start ($I \ll N$)

$\to S(t) \lesssim N$

$\to S(t) = N \quad \to \quad dI/dt = (\lambda - r) I$

$$I(t) = \exp((\lambda - r)t) = \exp\left(r\left(\lambda/r - 1\right)t\right)$$

$$\underbrace{R_0 \equiv \lambda/\tau}$$ → reproduction number = # of people infected by an individual before recovery

$$\left(\lambda/\tau\right)_c = 1$$

$R_0 > 1$ → exponential increase (active phase)

$R_0 < 1$ → exponential decrease (inactive phase)