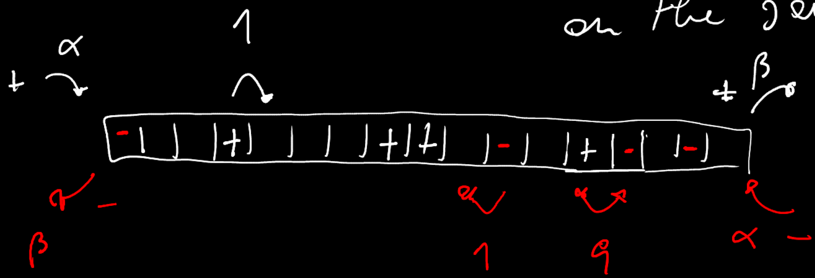
BRIDGE MODEL → TASEP with 2 particle types (+, −) moving in opposite directions on the same lane



open BC

Symmetry upon exchange + with −

if ($\beta \ll 1$; $\alpha, q \lesssim 1$): both + and − should stay in the HD phase

$\boxed{p + m \leq 1}$ whereas $p_{HD} + m_{HD} > 1$

densities of positive (p) and negative (m) particles

Spontaneous symmetry breaking!

PHASE DIAGRAM (in α, β plane)

MC ~ symmetric MC

LD-HD spontaneous symmetry breaking] $p < 1/2 < m$ (or $m < 1/2 < p$)

LD symmetric

LD-LD spontaneous symmetry breaking] → $p < m < 1/2$ (or $m < p < 1/2$)

SSB proved exactly for $\underline{\beta \ll 1 \; ; \; \alpha = q = 1}$

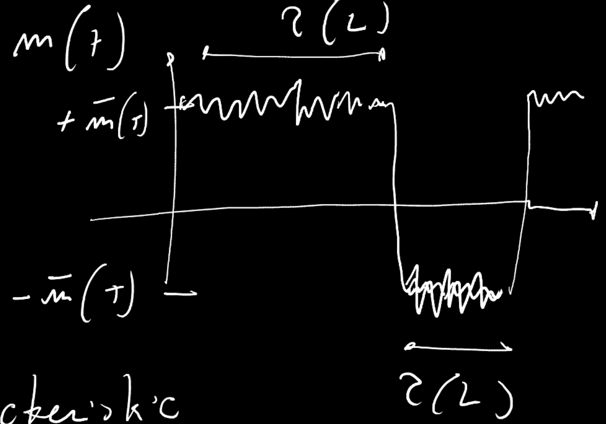
How can SSB be proven? Ising model in $d=2$

finite size L at $T < T_c$ $h = 0$

for $L \to +\infty$; expected values for magnetization
$\pm \bar{m}(T)$

if $L \gg \xi(T)$

correlation length



$\langle \tau(L) \rangle$ = characteristic time for magnetization inversion

$$\underbrace{\langle \tau(L) \rangle \sim \exp(L)}_{L \gg 1} \Rightarrow$$ Spontaneous symmetry breaking!

Ising model: switch on a symmetry breaking field $(h > 0)$ $\Rightarrow$ NO COEXISTENCE ANYMORE

Bridge model: switch on a field small enough $\Rightarrow$ COEXISTENCE

INFORMATION, PHYSICS & COMPUTATION

Entropy $\longrightarrow$ Information $\rightarrow$ { Data compression / Data transmission

X stochastic discrete variables assuming values $x \in \mathcal{X}$ (finite discrete set)

$\{p(x)\}_{x\in\mathcal{X}}$ PDF $p : \mathcal{X} \to [0,1]$

$\sum_{x\in\mathcal{X}} p(x) = 1$ expectation value for any real-valued function $f(X)$ $\mathbb{E} f = \sum_{x\in\mathcal{X}} p(x) f(x)$

$\mathrm{Var}\, f(X) = \mathbb{E}\{f^2(X)\} - \{\mathbb{E} f(X)\}^2$

$A \subseteq \mathcal{X}$ (subset)

$\mathbb{P}(X \in A) = \int_{x\in A} \overbrace{dp(x)}^{p(x)dx} = \int \mathbb{I}(x\in A)\, dp(x)$

indicator function $\underbrace{\mathbb{I}(S)}_{\text{statement}} = \begin{cases} 1 & \text{statement } S \text{ true} \\ 0 & S \text{ false} \end{cases}$

Gaussian PDF $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$

$\mathbb{E} X = m \; ; \; \mathrm{var}\, X = \mathbb{E}(x-m)^2 = \sigma^2$

<u>Jensen's inequality</u> $\mathcal{X} \subseteq \mathbb{R}$; f convex function

$[\forall x, y \; \forall \alpha \in [0,1] : f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)]$

$\Rightarrow \boxed{\mathbb{E} f(X) \geq f(\mathbb{E} X)}$

SHANNON ENTROPY H_X discrete random variable X with PDF $p(x)$

$$H_X \equiv - \sum_{x \in X} p(x) \log_2 p(x) = \mathbb{E} \log_2 \left[\frac{1}{p(x)} \right]$$

$\log_2 \longrightarrow$ Shannon entropy $\longrightarrow$ bits

entropy : measure of uncertainty $\rightarrow$ missing information

$H_X \approx \log_2$ (# of different values the variable may take)

Fair coin $\longrightarrow$ $H = 1 = \log_2 2$ $(N = 2)$

M fair coins ($N = 2^M$) $\longrightarrow$ $H = \log_2 N = M$ (M bits!)

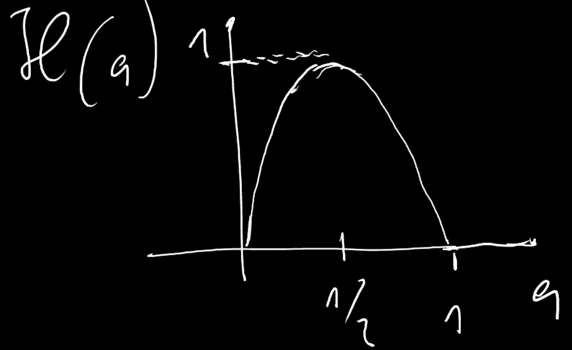
(entropy is additive)

fair dice (N faces) $\longrightarrow$ $H = \log_2 N$

"Unfair" coin ($q \neq 1/2$) (Bernoulli process): $p(0) = q$

$p(1) = 1 - q$

$$H_X = - q \log_2 q - (1-q) \log_2 (1-q) \equiv \mathcal{H}(q)$$



$q = 1/2 \longrightarrow \mathcal{H} = 1$ (fair coin)

$q = 0, 1 \longrightarrow \mathcal{H} = 0$ fixed result

Kullback-Leibler (KL) divergence between two PDFs $p(x)$, $q(x)$ defined over the same $\mathcal{X}$

$$D(q\|p) = \sum_{x\in\mathcal{X}} q(x)\log_2\left[\frac{q(x)}{p(x)}\right]$$

(i) $D(q\|p)$ is convex in q $\quad \left(\begin{matrix} 0\log 0 = 0 \\ 0 \log\left(\frac{0}{0}\right) = 0\end{matrix}\right)$

(ii) $D(q\|p) \geq 0$

(iii) $D(q\|p) = 0 \iff q(x) = p(x)$

↳ Jensen's with $-\log(x)$ is convex

$$-D(q\|p) = E_q \log_2\left[\frac{p(X)}{q(X)}\right] \leq \underbrace{\log_2 \overbrace{E_q\left[\frac{p(X)}{q(X)}\right]}^{=1}}_{=0}$$

$D(q\|p)$ ~ "distance" between q, p

$D(q\|p) \neq D(p\|q)$ (asymmetric)

Go back to Shannon entropy $H_X = -\sum_x p(x)\log_2 p(x)$

1) $H_X \geq 0$ 2) $H_X = 0 \iff X$ is certain ($p(x) = \delta_{x,x_0}$)

3) H_X is maximum when all N events are equiprobable $\bar{p}(x) = 1/N$

$\longrightarrow \quad H_x(\bar{p}) = \log_2 N \qquad$ generic $p(x)$

$$D(p \,\|\, \bar{p}) = \sum_x p(x) \log_2 [N p(x)]$$

$$= \log_2 N - H(p) \geq 0$$

$$\Rightarrow H(p) \leq \log_2 N = H(\bar{p})$$

(4) X, Y independent random variables

$$p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$$

$$H_{X,Y} = - \sum_{x,y} p(x) p(y) \log_2 [p(x) p(y)]$$

$$= H_X + H_Y$$ (additivity rule for independent events)

(5) any pair X, Y in general correlated

$\boxed{H_{X,Y} \leq H_X + H_Y} \leftarrow$ (proven from $D(p_{X,Y} \,\|\, p_X p_Y) \geq 0$)

correlation $\Rightarrow$ less entropy

$\Rightarrow$ more 'a priori' info on X, Y

⑥ Additivity rule for composite events

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \ ; \quad \mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$$

$$q_1 = \sum_{x \in \mathcal{X}_1} p(x) \quad (\text{prob. } \mathcal{X}_1)$$

$$q_1 + q_2 = 1$$

$$q_2 = \sum_{x \in \mathcal{X}_2} p(x) \quad (\text{prob. } \mathcal{X}_2)$$

$$r_1(x) = p(x)/q_1$$ (conditional prob. for x given $x \in \mathcal{X}_1$)

$$r_2(x) = p(x)/q_2$$ (" " for x given $x \in \mathcal{X}_2$)

$$\boxed{H_X = \mathcal{H}(q_1) + \tilde{H}(q_1, r)}$$

entropy related to subset choice

$$\mathcal{H}(q_1) = -q_1 \log q_1 - q_2 \log q_2$$

$$\tilde{H}(q_1, r) = -q_1 \sum_{x \in \mathcal{X}_1} r_1(x) \log_c r_1(x) - q_2 \sum_{x \in \mathcal{X}_2} r_2(x) \log_c r_2(x)$$

entropy related to event choice within subsets

$$\boxed{\text{generalized by chain rule for conditional entropies}}$$