LIFE DATA EPIDEMIOLOGY

Lecture 2: SIR model

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SIR model

- It is a **compartmental** model
- All N individuals of the population are categorized into one of 3 classes:
  - Susceptible (S): sane but without immunity
  - Infected (I): have the disease and spread it
  - Recovered (R): the disease ended and they are immune to it (or they are dead)
- Individuals can transit from S to I and from I to R, where they stay forever

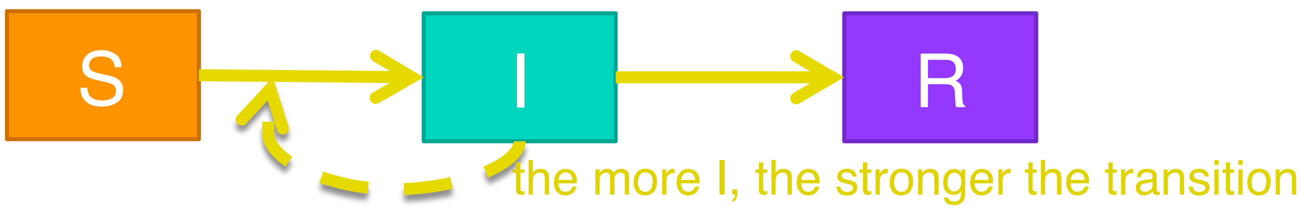
SIR model

- Individual-wise, the model is just a state-machine with two main transitions

- However, the model just considers the overall amount of individuals in each state
  - equal to S, I, R, respectively. $S + I + R = N$
  - take $s = S/N$, $x = I/N$, $r = R/N$: $s + x + r = 1$

Usual SIR model assumptions

- Homogeneous mixing: all people are in contact → contagion rate = βx, with $\beta > 0$



- Memoryless recovery: disease time is exponentially distributed with parameter μ → which implies $\mathbb{E}$[sickness duration] = $1/\mu$
- Closed system (no arrivals/departures)
- Deterministic (1st order approx) – while in reality there are elements of chance

SIR model equations

- Take $s = s(t)$, $x = x(t)$, $r = r(t)$, with t = time
- We can write the following equations

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\beta s x$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \beta s x - \mu x$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu x$$

all quantities >0
thus $s(t)$ decreases and $r(t)$ increases as t goes by

$\mathrm{d}s/\mathrm{d}t + \mathrm{d}x/\mathrm{d}t + \mathrm{d}r/\mathrm{d}t = 0$
because closed system

deterministic eq.s: no uncertainty

Known SIR approximations

- populations do not uniformly mix: model works well only for closed environments
- population is discrete: quantization error
- deterministic model: if *y* individuals are expected to change state, it is so
  - big N helps 2nd and 3rd points (law of large numbers) but is troublesome for the 1st
- duration of known diseases is ~fixed and not exponential - yet, simpler math

Basic reproductive ratio R_0

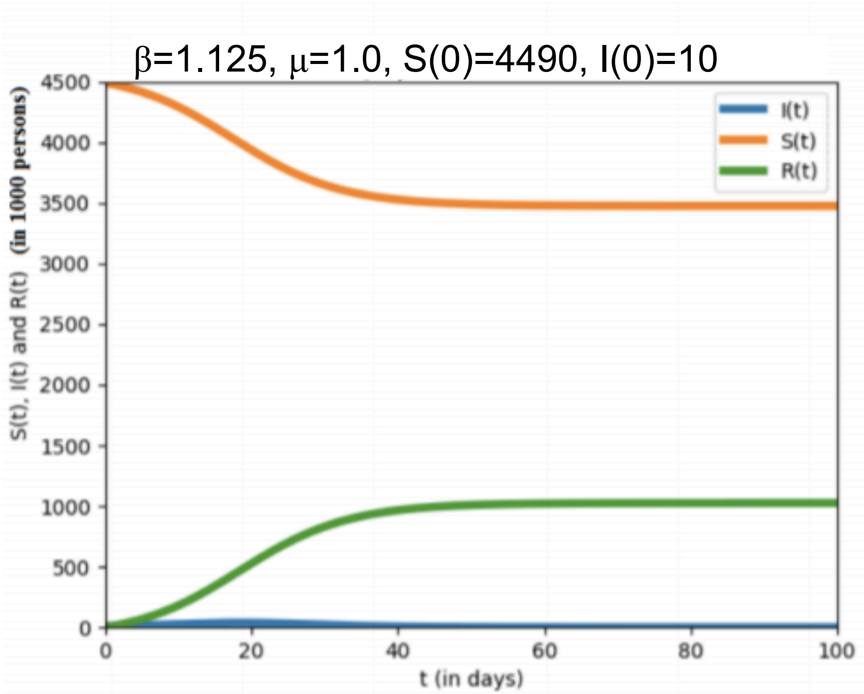
- Only susceptible individuals at the time 0: $s(0)=s_0\approx 1$ → spreading only if $x'(0) > 0$

$$\frac{dx}{dt} = \beta s x - \mu x > 0$$ → at $t=0$: $\beta/\mu > 1$

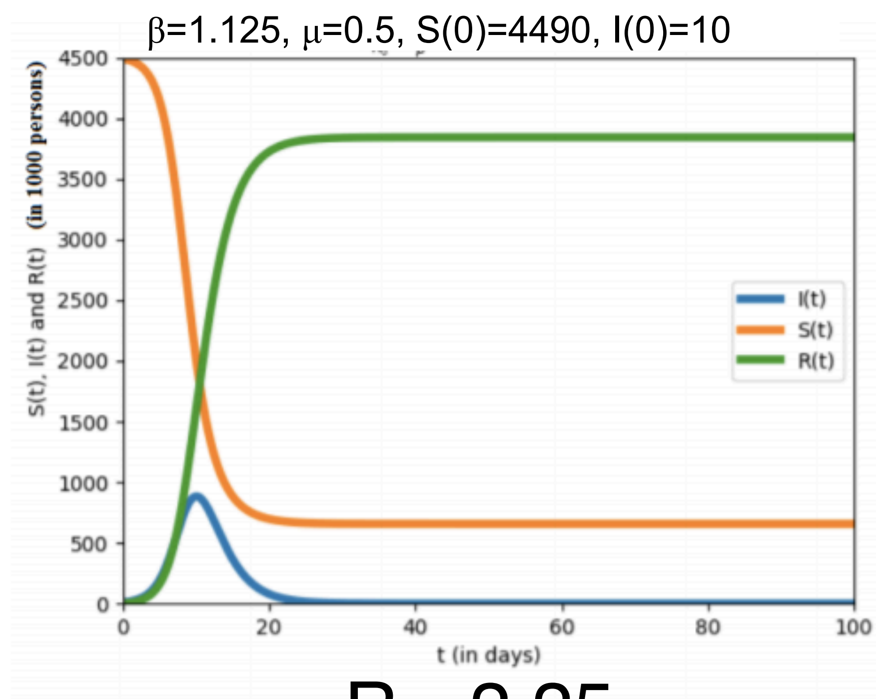
- The parameter $R_0 = \beta/\mu$ is called basic reproductive ratio and is often associated with the strength of the epidemics

Basic reproductive ratio R_0

- Numerical solutions of SIR show R_0's role



R_0=1.125



R_0=2.25

Basic reproductive ratio R_0

- Formally, $R_0 = \mathbb{E}$[#secondary infections caused by an infected individual at time 0]
  - estimates of R_0 available for various infections (albeit with high variability)

Disease	How	R_0	Disease	How	R_0
Measles	Airborne	12-18	Rubella	Airborne	5-7
Pertussis	Airborne	12-17	Mumps	Airborne	4-7
Diphtheria	Saliva	6-7	HIV/AIDS	Sexual	2-5
Smallpox	Contact	5-7	SARS	Airborne	2-5
Polio	Fecal-oral	5-7	Influenza'18	Airborne	2-3

Physical meaning of parameters

- How can we derive the parameters of the SIR model? If time measured in days:
  - $\mu = 1/\mathbb{E}$[#days spent being sick]
  - during its infectious days (typically $1/\mu$) an individual causes R_0 contagions (at most: true at time 0 when everybody else is S)

  → infection rate per infected = $R_0 / (1/\mu) = \beta$
  - β = rate of "contagious contacts" combining:
    p[contagion | contact] × p[contact]

Counteracting epidemics

- How to lower $R_0 = \beta/\mu$ for a weaker disease
- Either decrease β:
  - reduce contagion prob. in case of contact (e.g., no physical proximity, use protections)
  - reduce mixing probability (e.g. quarantine)
- Or increase μ (make recovery faster)
  - if infected individuals recover sooner, they have fewer chances to infect others

Herd immunity

- Or: vaccination! This implies that $s(0) < 1$
  - some individuals start in class R already
- Under the simplifying assumptions of SIR, the disease does not break out if vaccination rate is high enough (higher threshold)
  - recall initial spreading condition: $\beta s_0 - \mu > 0$
- But it does not require to vaccinate everybody (or does it? we will see)

Herd immunity

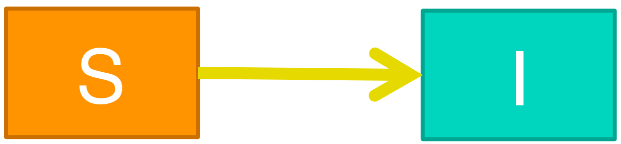
- Depending on R_0 one can compute the minimum required share of vaccinations
  - we need $r_0 = 1 - s_0 > 1 - \beta/\mu = 1 - R_0$
  - e.g., if $R_0 = 4.0$, >75% of the population should be vaccinated so $dx/dt < 0$ at $t = 0$
- Important disclaimer: true only under assumptions (homogeneous mixing)
  - in reality you need a much higher rate!

An application for game theory?

- Vaccinations require cooperation
  - the individual behavior of selfish players is not to vaccinate themselves (easier) and rely on herd immunity
  - however, then nobody vaccinates and there is no herd immunity whatsoever
- A prisoner / conspiracy theorist – dilemma!

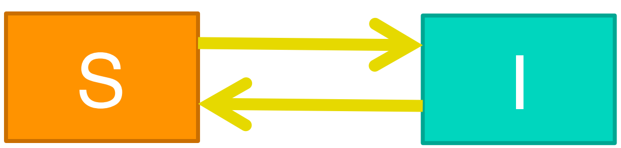
Model variations

- **SI model**
  - the disease is chronic, or cannot be healed



$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\beta s x, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \beta s x$$

- **SIS model**
  - healing = susceptible again (common cold)



$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}s}{\mathrm{d}t} = \beta s x - \mu x$$

Analytical solution of SI model

- We start with the SI model (simpler)
  - as a matter of fact, only one equation:

$$dx/dt=\beta s x \,, \qquad \text{where } s = 1-x$$

- Solve differential eq.: $dx\,/\,[\beta(1-x)x] = dt$

or: $1\,/\,[\beta(1-x)]\,dx + 1\,/\,[\beta x]\,dx = dt$

→ $-\log|1-x|\,/\,\beta \;+ \log|x|\,/\,\beta \;= t + C$

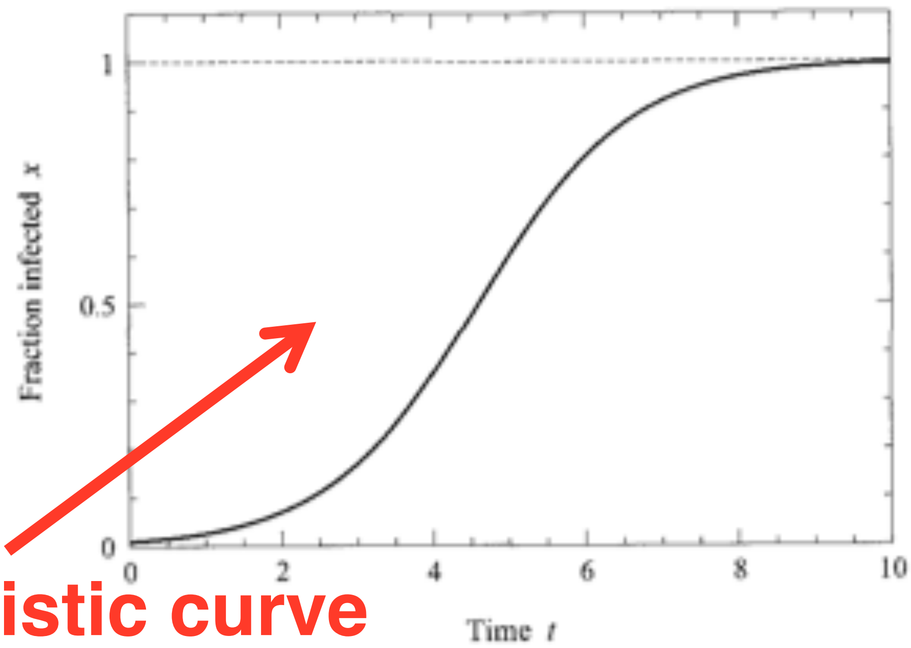
$$x/(1-x) = e^{\beta(t+C)} = A\,e^{\beta t} \quad \text{with } A = x_0/(1-x_0)$$

- Thus

$$x(t) = \frac{x_0\, e^{\beta t}}{1 - x_0 + x_0\, e^{\beta t}}$$

Analytical solution of SI model

- Solution $x(t) = \dfrac{x_0 \, e^{\beta t}}{1 - x_0 + x_0 \, e^{\beta t}}$ is a **sigmoid**
  - S-shaped function, starts as an exponential (almost all are S) but then saturates
  - from x_0 to 100% (consequence of SI assumptions)



called the **logistic curve**

Analytical solution of SIS model

- For the SIS model we can write

  $dx/dt = \beta(1-x)\,x - \mu x\,, \qquad \text{where } s = 1-x$

- Its solution is $x(t) = (1-\mu/\beta)\dfrac{C\,e^{(\beta-\mu)t}}{1+C\,e^{(\beta-\mu)t}}$

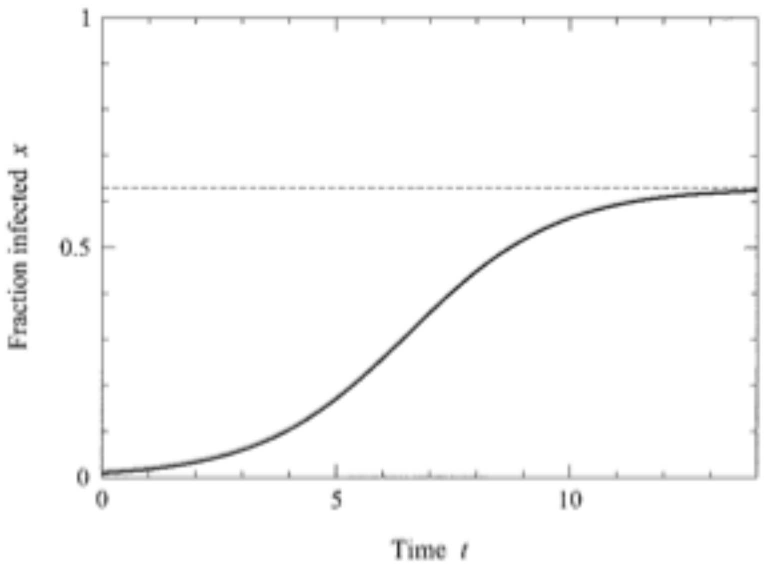
  with int.constant $C = \beta\, x_0 / (\beta - \mu - \beta\, x_0)$

  can be omitted as x_0 is typically small

- This results in $x(t) = x_0 \dfrac{(\beta-\mu)e^{(\beta-\mu)t}}{\beta-\mu+\beta x_0\, e^{(\beta-\mu)t}}$

Analytical solution of SIS model

- The logistic curve is again a **sigmoid** but its limit is $(\beta-\mu)/\beta$ as opposed to 1 of SI
  - It is also visible that the infection dies out if $\beta < \mu$ as seen for SIR
  - For $\beta > \mu$: an endemic steady-state is the saturation point, at which, on average, #cured = #contagions



SIR model

- This model cannot be fully solved in closed form (previous plots are numerical)
- We can start from SIR equations (1)+(3)

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -\beta s x$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu x$$

- deriving x yields

$$\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{\beta}{\mu}\frac{\mathrm{d}r}{\mathrm{d}t}$$

- and integrating: (if we set $r_0 = 0$)

$$s = s_0\, \mathrm{e}^{-R_0 r}$$

- Thus, s exponentially decays in r

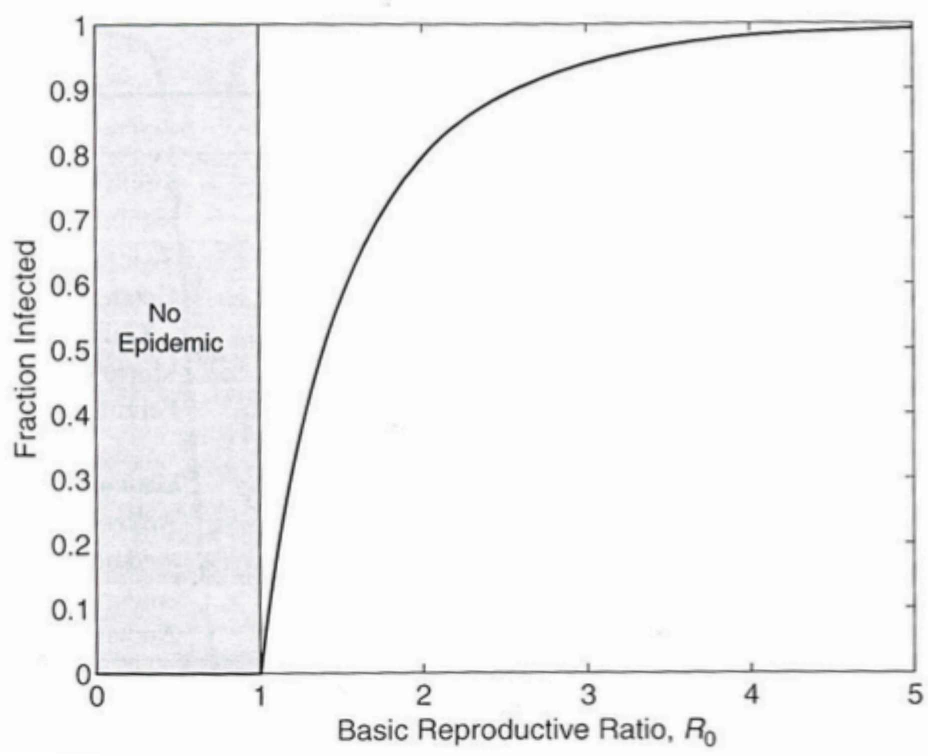
SIR model: asymptotic regime

- A remark on equation $s = s_0\, e^{-R_0 r}$
  - r increases with t but the exponential term never goes to 0 → however contagious the disease, there are individuals avoiding it (note: true under the limits of quantization)
  - the reason is that in the SIR model, the infected individuals cease to be contagious → the disease extinguishes as x goes to 0

SIR model: asymptotic regime

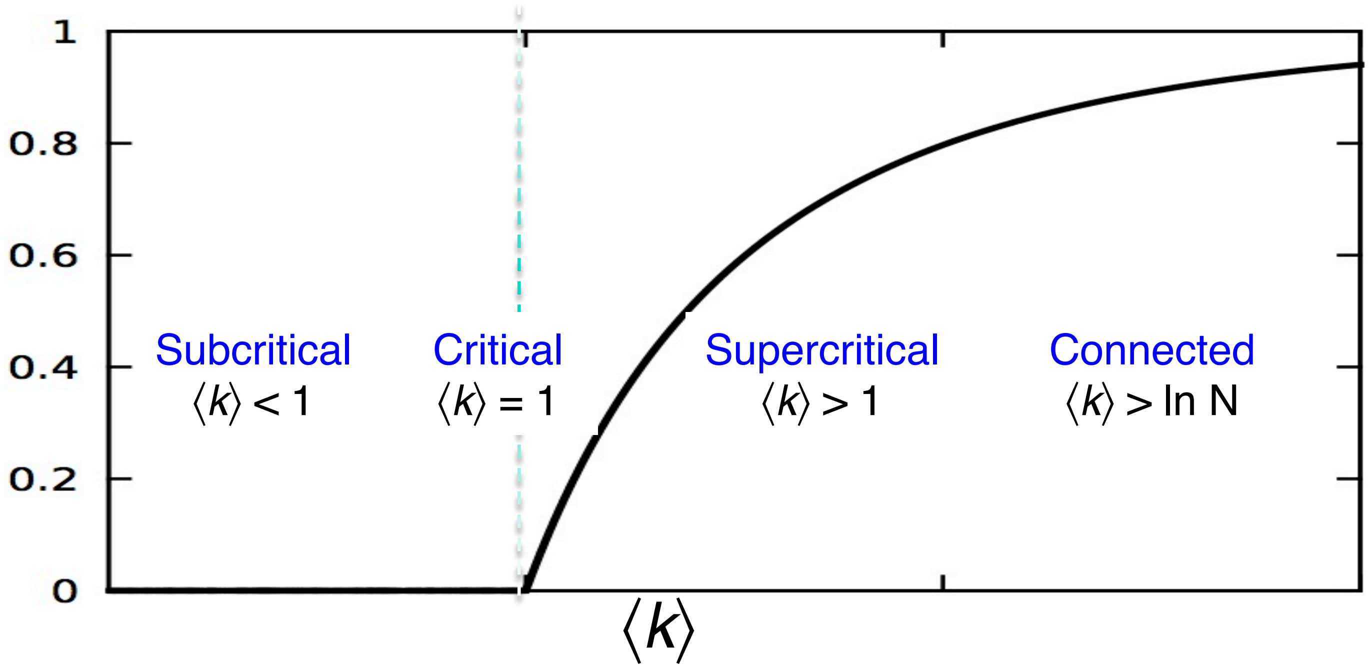
- If $s = s_0 \, e^{-R_0 r}$, the fraction of individuals avoiding the disease must be > e^{-R_0}
  - of course if $R_0 > 1$

- Asymptotic share of susceptible is depending on R_0



Connection with random graphs

- Whatever the expression of R_0, threshold criterion $R_0>1$ has a random graph interpretation: $\langle k \rangle>1$ for a GC to appear
  - same graph for |GC| and r_∞ of SIR



SIR model: asymptotic regime

- At $t = \infty$ we only have this fraction of individuals spared by the epidemics +the others (now recovered): x_∞ must be 0

$$s_\infty + r_\infty = s_0\, e^{-R_0\, r_\infty} + r_\infty = 1$$

- This last equation is relevant since r_∞ gives the share of the population that contracted the disease at any time
  - we assumed $r_0 = 0$ and we have $x_\infty = 0$

SIR model: asymptotic regime

- The asymptotic values can be used to characterize SIR parameters
  - Asymptotic equation $s_0\, e^{-R_0 r_\infty} + r_\infty = 1$ relates R_0 with the fraction of individuals that ever got infected at one point
  - Other relationships are possible including non-asymptotic case where $x(t) \neq 0$
- Especially, characterizing R_0 gives a rough idea of the "infection strength"

Importance of R_0 so far

- Threshold behavior: spreading if $R_0 > 1$
- Vaccination rate is $1-1/R_0$
  - under homogeneous mixing
- Initial trend: exponential with coefficient R_0
  - this can be useful for estimates
- %spared depends on it (must be $> e^{-R_0}$)
  - this is actually easier to estimate ex-post

- More when we introduce **demography**!