Exercises

Thursday, October 24, 2019 2:26 PM

Exercise 1 A contagious disease is spreading over a continent. The disease causes heavy migraines but no other symptoms; it has no known cure and, if contracted, is permanently affecting the individuals for their entire life. At day 0, an island with population of exactly 10000 individuals is quarantined and isolated from the continent to avoid that the disease spreads there. Unfortunately, two inhabitants of the island were already infected with the disease. Contacts on the island are frequent among all individuals and happen with identical probabilities among all pairs of inhabitants (regardless of their status of infected or not). Every inhabitant has an average of 15 contacts per day, each of them resulting in a 2% probability of contracting the disease if the contact involves an infected individual meeting another who is unaffected.

1. Detail what kind of compartmental model is appropriate for the disease, and what parameters characterize its equations. Ignore any demography.

2. How many infected individuals do you expect at day 24?

$$S \xrightarrow{\beta} I$$

SI model

$$\frac{dx}{dt} = \beta s x \qquad \frac{ds}{dt} = -\beta s x$$

$N = 10000$ $\quad \beta = 0.3 \text{ days}^{-1}$

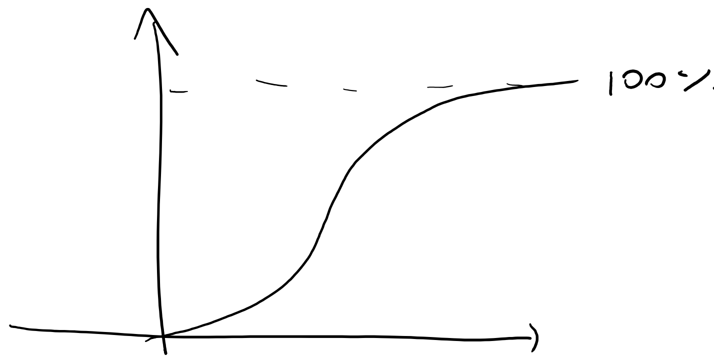
$X(1)$ $\quad X(0) = 2$ $\quad x_0 = 0.0002 = \frac{X(0)}{N}$

$$X(1) = X(0) + \beta \underbrace{s_0}_{\simeq 1} \underbrace{\Delta t}_{=1 \text{ day}}$$

~~demography~~ check: lifespan = 50 years

$\lambda \simeq 0.0001$

We know that the solution of the ODE is a sigmoid



$$x(t) = \frac{x_0 \, e^{\beta t}}{1 + x_0 \, e^{\beta t}}$$

$t = 24$ days

$$x(24) = 21.13\%$$

$$X(24) = 2113 \text{ people}$$

SIDE QUESTION: When is the disease spread over the entirety of the island?

$\cancel{x(t) = 1}$ $\qquad x(t) > 0.9999$

$$0.0001 \, x_0 \, e^{\beta t} > 0.9999$$

$$t > \left(\ln \frac{0.9999}{0.0001 x_0}\right) / \beta = 59{,}09$$

3. A cure is found when 50% of the population of the island is affected. It requires an average of N days of therapy but eventually gives full recovery and permanent immunity. All infected individuals immediately start being treated as soon as they feel the symptoms, but while undergoing the therapy they are still contagious. In the end, it is found that the disease is eradicated with 10% of the population having never contracted it. How long is N_{terapy}?

~~SI~~

$$S \xrightarrow{\beta} I \xrightarrow{\mu} R$$

$\mu = 1/N_{\text{terapy}}$

SIR does not have a closed-form solution

$$s_0 e^{-R_0 r_\infty} + r_\infty = 1$$

$s_0 = 50\%$

$R_0 = ?$

$s_\infty = 10\%$ $\quad r_\infty = 90\%$

$$0.5 \exp(-R_0 \cdot 0.9) = 0.1$$

$$-R_0 \cdot 0.9 = \ln 1/5$$

$$R_0 = \frac{\ln 5}{0.9} = 1.7883 = \beta/\mu$$

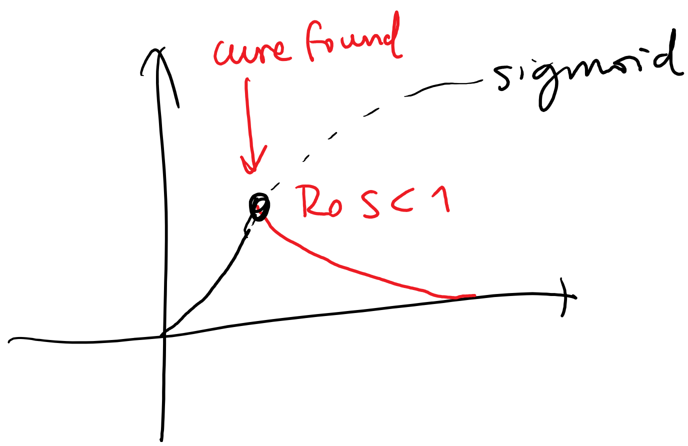
$$N_{\text{terapy}} = 1/\mu = 5.96 \text{ days}$$

Also check that at day 0 after finding the therapy $s_0 = 0.5$ $R_0 = 1.7883$

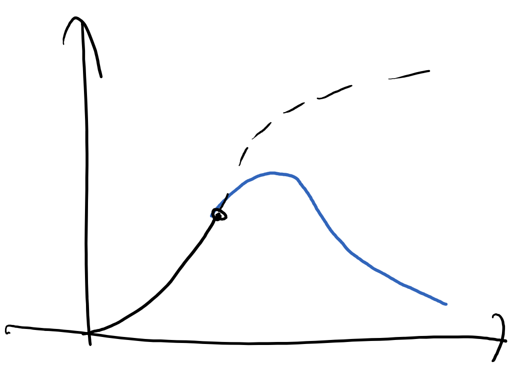
$$\frac{dx(t)}{dt} = \beta s x - \mu x$$

derivative is positive if $R_0 s > 1$

but $s_0 = 0.5$



if $R_0 s > 1$



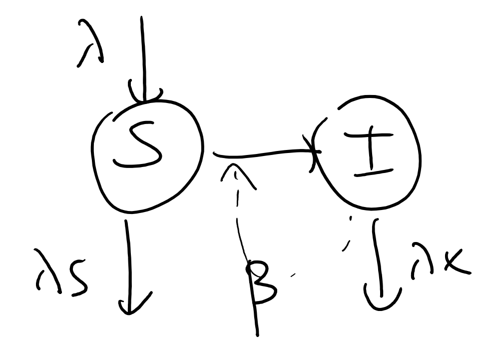
Exercise 2 These are some relevant data about the country of Bravonia.

GDP per capita	300 EUR / year
annual GDP change	5.5%
population	11620
annual population change	0.0%
average life expectancy	60 years

In Bravonia, the most used transportation means is the bicycle. Once you learn how to ride a bicycle, you never forget it. Also, you can teach other people how to ride a bicycle (assume you can even teach a newborn child, i.e., the time when a newborn is still immune to learning is negligible). The average inhabitant of Bravonia that knows how to ride a bike tries teaching 5 other people every 10 years. However, not all inhabitants learn how to ride a bike from their first teacher, this process has a success rate of 27% (independent for every new teacher). Assume that the teaching process, if successful, is instantaneous, i.e., you learn immediately how to ride a bike.

1. How can this be characterized by an epidemic model? Write down the equations and quantify the relevant parameters. Assume homogeneous mixing.
2. How many people in Bravonia do not know how to ride a bike?
3. What is the average age when people in Bravonia learn how to ride a bike?

Time measured in <u>YEARS</u>



with demography

$\lambda = 1/60 \text{ years}^{-1}$

$\beta = 0.135 \text{ years}^{-1}$

$$\frac{ds}{dt} = \lambda(1-s) - \beta s x$$

$$\frac{dx}{dt} = \beta s x - \lambda x$$

$$\frac{dx}{dt} = \beta s x - \lambda x$$

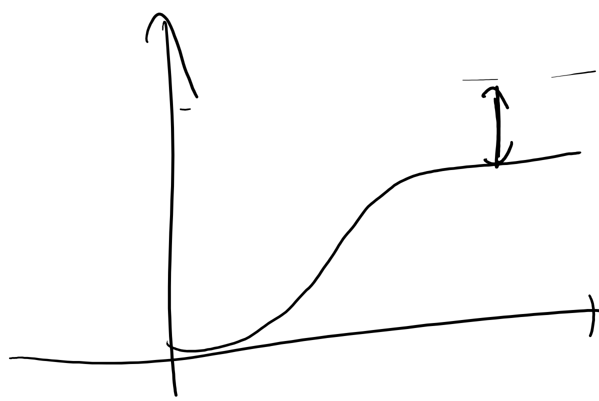
This is analogous to SIS

(even though there is no transition $I \to S$

but the same number of individuals

exit I as the "net amount" of those

entering S



$$R_0 = \frac{\beta}{\lambda + \mu} = 8.1$$

$$s_\infty = 1/R_0 = 12.34\%$$

$$S_\infty = s_\infty \cdot N = 1434 \text{ people}$$

A = average age at 1st infection

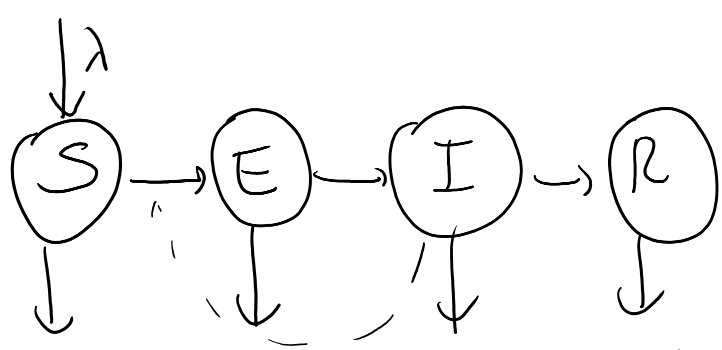
$$= \frac{1}{(R_0 - 1)\lambda} = 8.45 \text{ (years)}$$

Exercise 3 The ILLUMINATI are a secret society in Italy. Only Italian citizens of 18 years or above, of both genders and any social census, may join the ILLUMINATI.and to do so, they have to accomplish two steps. First, if you desire to become an ILLUMINATO, you must spend 2 years studying the tenets, the rituals, and the code of honor of the society. After that, you spend 5 years recruiting new aspiring members, following which, you just are a full-fledged ILLUMINATO for life. Nobody leaves the way of the ILLUMINATI after they start studying the tenets, until they die; however, full-fledged members just keep quiet about their ILLUMINATI status and do not recruit new affiliates. During the recruitment phase, members approach on average 3 people a year, assumed to be homogeneously mixed, and each of them accepts to enter the ILLUMINATI with a probability of 7.5%. In Italy, there are 50 million adult citizens of more than 18 years of age and the average life expectancy is 80 years.

1. Describe this system with a proper epidemic model, writing down the parameters.

2. Show that the ILLUMINATI are endemic within the Italian society.

3. Compute how many full-fledged ILLUMINATI are there in Italy.



TOTAL RATE λ

time is in YEARS

$\lambda = 1/62 \text{ years}^{-1}$

$\sigma = 1/2 \text{ years}^{-1}$

$\mu = 1/5$ years^{-1}

$\beta = 7.5\%$ of $3 = 0.225$ years^{-1}

$N = 5 \cdot 10^7$

endemic means $R_0 > 1$

In SEIR $R_0 = \dfrac{\beta\sigma}{(\lambda+\mu)(\lambda+\sigma)} = 1.0087044$

Asymptotic shares

$s_\infty = 1/R_0 = 99.137\%$

$x_\infty = (R_0 - 1)\lambda/\beta = 0.0623\%$

$y_\infty = (R_0-1)\lambda(\lambda+\mu)/(\sigma\beta) = 0.0269\%$

$r_\infty = 1 - s_\infty - x_\infty - y_\infty = 0.7737\%$

↑ full-fledged ILLUMINATI

$R_\infty \simeq 386000$ people

Exercise 4 The "#aweekofkisses" challenge is spreading over social media. You send a video to your friends a video where you kiss the air, with a smoochie, to promote peace. As the name implies, participants to the challenge keep sending videos for one week, contacting an average of $Q = 40$ people during that entire time. Assume these contacts are uniformly spread over the week. Not everyone who is contacted decides to propagate the challenge, though. It has been found that the rate of adoption is significantly higher if the sender and the receiver are of opposite genders. Indeed, the probability of a person adopting it from a sender of the opposite gender is $a = 10\%$, while the probability of a female adopting the challenge from another female is $b = 1\%$ and the probability of a male adopting the challenge from another male is $c = 0.2\%$. Assume that the social media have perfect $50-50$ split of genders in their population, and contacts are gender-neutral, so people are equally likely to have contacts of theirs or the opposite gender.

1. What model can you use for this spreading? Write its WAIFW matrix.
2. Does this "#aweekofkisses" challenge spread? What parameter can you use to characterize this?

SIR with risk structure

H = female L = male

$n_H = n_L = 0.5$ $a = 10\%$ $b = 1\%$ $c = 0.2\%$

Time measured in DAYS $\mu = 1/7$

WAIFW $\beta = \begin{bmatrix} \beta_{HH} & \beta_{HL} \end{bmatrix} = \begin{bmatrix} \frac{40}{7} b & \frac{40}{7} a \end{bmatrix}$

WAIFW $\beta = \begin{bmatrix} \beta_{HH} & \beta_{HL} \\ \beta_{LH} & \beta_{LL} \end{bmatrix} = \begin{bmatrix} \frac{40}{7} b & \frac{\text{[illegible]}}{7} a \\ \frac{40}{7} a & \frac{40}{7} c \end{bmatrix}$

$$R_0^H = \frac{\beta_{HH} + \beta_{HL}}{\mu} = 2.2$$

$$R_0^L = \frac{\beta_{LH} + \beta_{LL}}{\mu} = 0.24$$

These values are useful just at the beginning (invasion) but then the dynamics become slowed

$\Rightarrow$ we need the eigenvalues of the Jacobian

(time is out: complete the exercise by yourself as homework :)