Difference between eq. states and non-equilibrium stationary states (NESS)

Master eq. (continuous time, discrete set of microstates $\{s_i\}$)

$$\frac{\partial p(s,t)}{\partial t} = \sum_{s'} \left[p(s',t) W_{s,s'} - p(s,t) W_{s',s} \right]$$

at stationarity: $\frac{\partial p}{\partial t} = 0 \Rightarrow \boxed{\sum_{s'} \left[p^s(s') W_{s',s} - p^s(s) W_{s',s} \right] = 0}$

steady stationary states $\Longleftrightarrow$

Stronger (equilibrium) condition:

DETAILED BALANCE: $\boxed{p^{eq}(s') W_{s',s} = p^{eq}(s) W_{s,s'} \quad \forall s,s'}$

equilibrium states $\Longleftrightarrow$

<u>NESS</u>: stationary states <u>without</u> <u>DB</u>

- Equilibrium: (T fixed → Boltzmann distribution $p^{eq}(s) = \frac{\exp(-\beta E(s))}{Z}$)

→ find transition rates that satisfy DB with $p^{eq}(s)$

$$\frac{W_{s',s}}{W_{s,s'}} = \frac{p^{eq}(s')}{p^{eq}(s')} = \exp\left(-\beta\left[E(s') - E(s)\right]\right)$$

Montecarlo simulation → define $W_{s',s}$ to sample $p^{eq}(s)$

Metropolis algorithm $W_{s',s} = \min\left[1, \exp(-\beta \Delta E)\right]$

• NESS — no "à priori" knowledge of the stationary prob. distribution $p^S(s)$

(no general principles like Boltzmann)

Assign rates $W_{s',s}$ to define the model

→ use stationarity condition to find $p^S(s)$

Problem: given a set of rates $\{W_{s',s}\}$: are they compatible with DB? (without knowing $p^{eq}(s)$)

$\boxed{\text{THEOREM}}$ $\{W_{s',s}\}$ satisfies DB for $p^{eq}(s)$ → $\left[\dfrac{W_{s',s}}{W_{s,s'}} = \dfrac{p^{eq}(s')}{p^{eq}(s)} \quad \forall s, s' \right]$

$$\Updownarrow$$

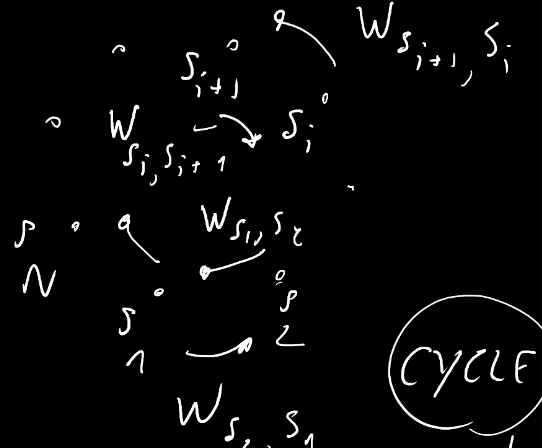
$$\underbrace{\prod_{i=1}^{N} W_{s_{i+1},s_i} = \prod_{i=1}^{N} W_{s_i,s_{i+1}}}$$

$\forall$ set $\{s_1, \dots, s_N\}$ of microstates

reversibility condition on a cycle

(PBC) → $s_{N+1} = s_1$

→ NO CURRENT



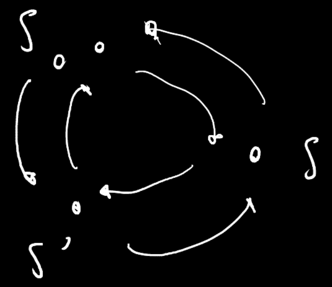
$\boxed{\text{PROOF}}$ $\Downarrow$ trivial $\displaystyle\prod_{i=1}^{N} \frac{W_{s_{i+1},s_i}}{W_{s_i,s_{i+1}}} = \prod_{i=1}^{N} \frac{p^{eq}(s_{i+1})}{p^{eq}(s_i)} = 1$

$$\Uparrow \quad \underbrace{p^{eq}(s) = p^{eq}(s_0)}\,\frac{W_{s,s_0}}{W_{s_0,s}} \quad \forall s \rightarrow \left(\begin{array}{c} p^{eq}(s_0) \\ \text{from} \\ \text{normalization} \end{array}\right)$$

OB holds for s, s_0 $\forall s$

(for a given "reference" state s_0)

OB $\forall s', s$ $(s', s \neq s_0)$



reversibility condition

$$W_{s,s_0} W_{s',s} W_{s_0,s'} \overset{\text{"}}{=} W_{s',s_0} W_{s,s'} W_{s_0,s}$$

$$\underbrace{\frac{W_{s,s_0}}{W_{s_0,s}}}_{\overset{\text{"}}{} \frac{p^{eq}(s)}{\cancel{p^{eq}(s_0)}}} W_{s',s} = \underbrace{\frac{W_{s',s_0}}{W_{s_0,s'}}}_{\frac{p^{eq}(s')}{\cancel{p^{eq}(s_0)}}} W_{s,s'} \rightarrow \underbrace{\frac{W_{s',s}}{W_{s,s'}} = \frac{p^{eq}(s')}{p^{eq}(s)}}$$

OB holds $\forall s, s'$

C.V.D

NON EQUILIBRIUM PHASE TRANSITIONS

- TRANSITIONS WITH ABSORBING STATES
- TRANSITIONS IN DRIVEN SYSTEMS (Directed percolation)

$\boxed{\text{SYSTEMS WITH ABSORBING STATES}}$

Absorbing state: state s such that $W_{s,s} = 1$

$\Rightarrow \underbrace{W_{s',s} = 0 \ \forall s' \neq s}$

DB would imply $\underbrace{p^{eq}(s') = 0 \ \forall s' \neq s}$

$\Rightarrow$ Transition to absorbing (inactive) phase
takes place out-of-equilibrium

DIRECTED PERCOLATION UNIVERSALITY CLASS

(one absorbing phase)

fluid through porous medium (filter coffee, oil from rocks, electric current)

heterogeneous → stochastic activation of "percolation channel" with probability p $(0 < p < 1)$

Isotropic percolation

square lattice; site
- wet (active)
- dry (inactive)

wet sites can wet neighbouring sites through active bonds (pores)

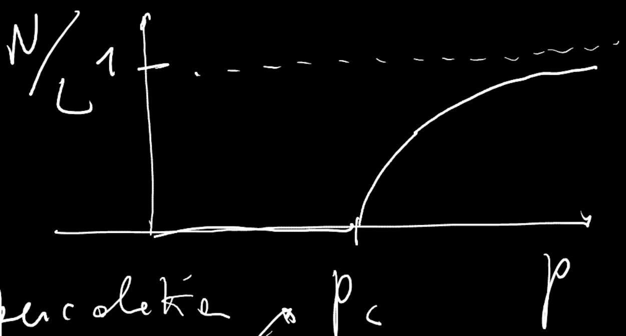
bonds are active with prob. p → bond percolation
(all sites are "potentially" active)

sites are active with prob. p → site percolation
(all bonds are "potentially" active)

is there a percolating "spanning" cluster
↳ from one side to the other of the system

N = size of the largest active cluster (# of sites connected by active bonds) → active cluster size

L = system size

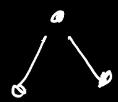


percolation threshold → p_c

infinite clusters appear for $p \geq p_c$ ($L \to \infty$)

at $p = p_c$ (critical point) the percolating cluster is <u>fractal</u>

So far: Isotropic percolation → equilibrium

<u>DIRECTED PERCOLATION</u> (only  <u>not</u> )

flow direction = <u>time direction</u>

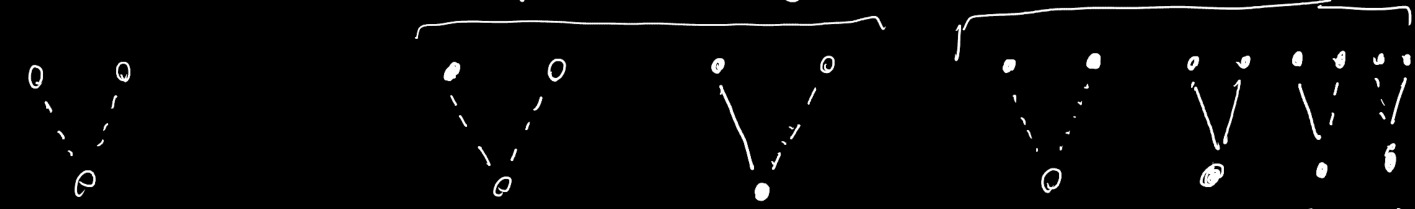
DP in $d+1$ = non equilibrium process in d

DP = phase transition from fluctuating active phase ($p > p_c$) to an <u>inactive absorbing phase</u> ($p < p_c$) ↳ (all sites are dry / inactive)

Possible dynamical processes (stochastic rules)

t

$t+1$



probability 1 (absorbing state) $\quad 1-p \quad p \quad (1-p)^2 \quad 1-(1-p)^2 = p + (1-p)p$

$$S_i(t) = \begin{cases} 1 & \text{site } i \text{ is active at time } t \\ 0 & \text{site } i \text{ is inactive at time } t \end{cases} \qquad 1 \le i \le L$$

$N(t) = \sum_i S_i(t)$ # of active sites at time t

typical initial condition: $S_i(0) = \delta_{i,i_0}$ (one active initial site)

$$\Rightarrow \underbrace{0 \le N(t) \le t+1}$$

p_c = percolation threshold

$p < p_c$ → subcritical inactive phase

No infinite clusters ← $\langle N(t) \rangle$ average over different realizations

$$\langle N(t) \rangle \underset{t \gg 1}{\sim} \exp(-t/\xi_\parallel) \xrightarrow[t \to \infty]{} 0$$

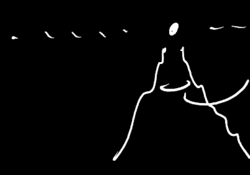
$p > p_c$ → supercritical active phase

infinite clusters exist with finite probability

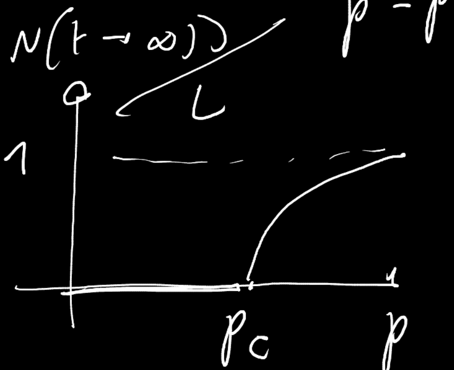
$$\langle N(t) \rangle \underset{t \gg 1}{\sim} t$$

(for infinite size system)

finite density of active sites



cone opening angle depends on $p - p_c$



$p = p_c$ at the percolation threshold

$$\underbrace{\langle N(t) \rangle \underset{t \gg 1}{\sim} t^{\theta}} \quad \text{(for infinite size)}$$

clusters of all sizes (fractal clusters) scale free behaviour and power-laws

critical exponent (θ)

<u>NO</u> exact solution for DP in 1 + 1 (d = 1)

Knowledge from numerical evidence $p_c \simeq 0.645$ non-universal

$\underbrace{\vartheta \simeq 0.302}$ universal