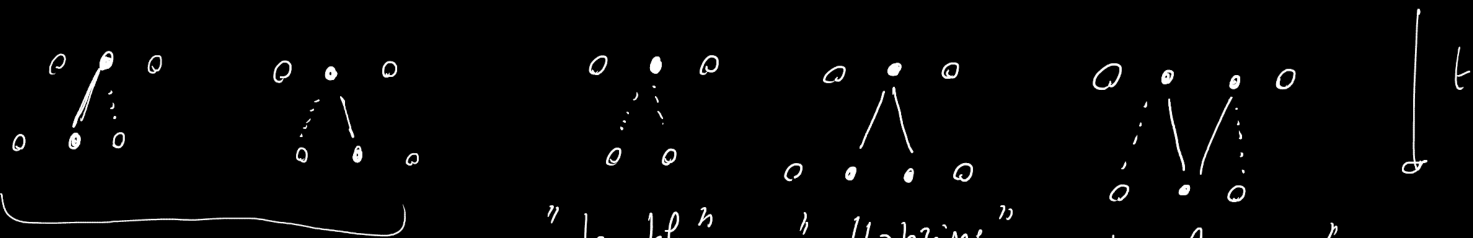
Connection of DP with REACTION-DIFFUSION



diffusion

$A \to A$

"death"

$A \to \emptyset$

"offspring"

$A \to 2A$

"coalescence"

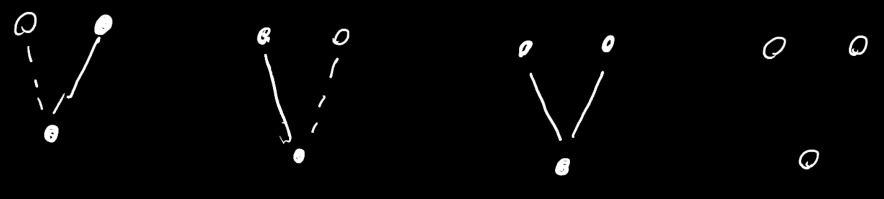
$2A \to A$

Generalization of DP: DOMANY-KINZEL MODEL

(example of CELLULAR AUTOMATA) — sites updated in parallel

CA = lattice, binary (active/inactive), local, synchronous, Markovian

- binary → exclusion process (no double occupancy)
- local → evolution rules depend on neighboring sites
- synchronous → sites updated in parallel
- Markovian → $t+1$ depends only on t



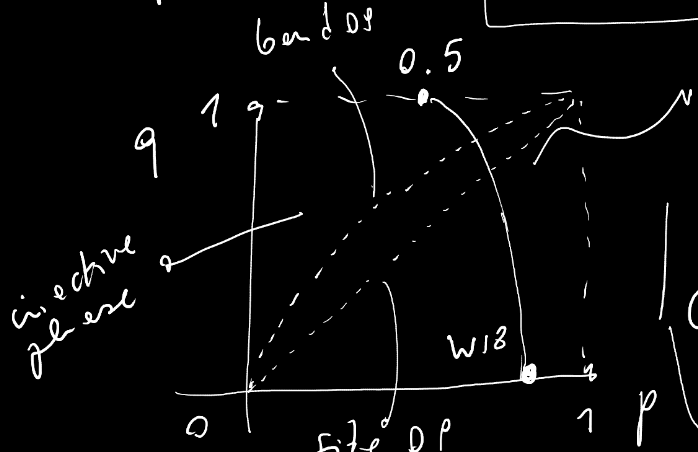
p $\quad$ p $\quad$ q $\quad$ 1

critical line

$p_c(q)$

for $0 \leq q < 1$ → same universality class of DP

Phase diagram in (p, q) plane



$q = 1 \to p_c = 1/2$

Compact Directed percolation (CDP)

1) bond DP: $q = p(2-p)$

2) site DP: $q = p$

3) $q = 0$ NO COALESCENCE

W18

4) $q = 1$ CDP $\underline{\underline{p_c = 1/2}}$ ("symmetry" between active and inactive sites)

active spots $\circ\,\underbrace{\bullet\bullet\bullet\bullet}_{\text{size } 4}\,\circ\,\circ$ $\qquad$ $\circ\,\underbrace{\bullet\bullet}_{\text{size } 2}\,\circ\,\underbrace{\bullet\bullet\bullet}_{\text{size } 3}\,\circ$

Phase transitions → order parameters, correlation length and critical exponents

control parameter $p \longrightarrow T$ (in the Ising model)

(p, q)

order parameter

$$m = \frac{\langle \sum_i S_i \rangle}{L}$$

thermodyn. ensemble average

one initially active site $N(t) = \langle \sum_i S_i(t) \rangle$ ⇝ average over different realizations of a stochastic process

homogeneous initial conditions (fixed fraction of initially active sites)

$$\rho(t) = \frac{N(t)}{L}$$

"survival probability" $\underline{P(t)} = \langle 1 - \underbrace{\prod_{i=1}^{L} (1 - S_i(t))} \rangle$

for a trajectory with one initially active state

0 all empty sites

1 any other state with at least 1 active site

fraction of "survived" trajectories

Order in the limits: $\underbrace{\text{first } L \to \infty, \text{ then } t \to \infty}$

$\rho(t)$, $P(t) = 0$ in the inactive phase; > 0 in the active phase

Critical exponents $\rho(\infty) \sim (p - p_c)^{\beta}$, $\longrightarrow m \sim (T_c - T)^{\beta}$

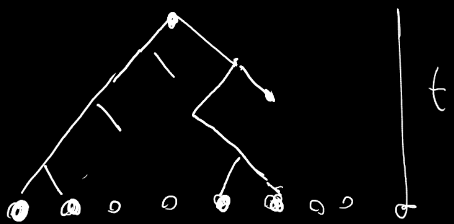
in general $P(\infty) \sim (p - p_c)^{\beta'}$

$\beta \neq \beta'$ $\left(\begin{array}{ll} q = 1 \;; \text{CDP} & \Rightarrow \beta \neq \beta' \\ q < 1 \;; \text{DP} & \Rightarrow \beta = \beta' \end{array} \right)$

$\beta = \beta' \simeq 0.276$
$(d = 1)$

Bond DP $\Rightarrow$ $\underline{P(t) = \rho(t)}$ — because of time reversal symmetry

$P(t)$



fraction of active sites on top (having started with all initially active sites on bottom)

fraction of survived trajectories from top = (having started with all initially active sites on bottom)

average over different positions of the initially active site = average over different realizations of a stochastic process

$\boxed{\text{self-averaging}}$ $\longrightarrow$ IN THE THERMODYNAMIC LIMIT

Also for other examples of DP

$\beta = \beta'$ because $P(t) \underset{t \gg 1}{\sim} \rho(t)$ (numerical evidence)

Non equilibrium: two independent correlation lengths spatial $\xi_\perp$ and temporal $\xi_\parallel$

$$C_{|i-j|} = \left\langle \lim_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t} \left(s_i(\tau)-\bar{s}\right)\left(s_j(\tau)-\bar{s}\right)\right\rangle$$

$$\bar{s} = \lim_{t\to\infty}\frac{1}{t} \sum_{\tau=0}^{t} \langle s_i(\tau)\rangle$$ does not depend ~~on~~ i $(t\to\infty)$

$$C_{|i-j|} \underset{|j-i|\gg 1}{\sim} \exp\left(-\frac{|i-j|}{\underbrace{\xi_\perp}_{\text{spatial correlation length}}}\right)$$

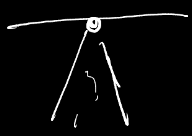
$$C(t) = \left\langle \lim_{L\to\infty} \frac{1}{L}\sum_{i=1}^{L} \left(s_i(t)-\bar{s}\right)\left(s_i(0)-\bar{s}\right)\right\rangle$$

$$C_t(t) \underset{t\gg 1}{\sim} \exp\left(-t/\underbrace{\xi_\parallel}_{\text{temporal correlation length}}\right)$$

$p < p_c$ $\quad \xi_\perp$ typical size, $\xi_\parallel$ typical time of clusters before they disappear

$p > p_c$ $\quad \xi_\perp$ typical size, $\xi_\parallel$ typical duration time of islands of empty sites within clusters

$\xi_\perp / \xi_\parallel$ = cone opening



with one initially active site

$p \to p_c$ $\quad \xi_\perp, \xi_\parallel$ diverges (either $p \nearrow p_c$ and $p \searrow p_c$)

$$\xi_\perp \sim |p-p_c|^{-\nu_\perp}, \quad \xi_\parallel \sim |p-p_c|^{-\nu_\parallel}$$

$$\boxed{z = \nu_\parallel / \nu_\perp} \rightarrow \text{dynamical exponent}$$

$$N(t) \sim t^{\theta} \quad (p = p_c)$$

$$\underbrace{\vartheta = \frac{d\nu_\perp - \beta - \beta'}{\nu_\parallel}}$$ Scaling laws: d = spatial dimension

PHENOMENOLOGICAL SCALING THEORY

$\Delta = |p - p_c|$ (distance from critical point)

→ properties of DP process are invariant under a (scaling transformation):

$\Delta \to \Lambda\Delta$; $x \to \Lambda^{-\nu_\perp} x$; $t \to \Lambda^{-\nu_\parallel} t$; $\rho \to \Lambda^{\beta}\rho$; $P \to \Lambda^{\beta'} P$

find $\rho(t)$ ($p = p_c$)

$$\rho(\Lambda^{-\nu_\parallel} t) = \Lambda^{\beta}\rho(t) \to \Lambda^{-\nu_\parallel} t = 1$$

$$\Lambda = t^{1/\nu_\parallel}$$

$$\to \rho(t) = \Lambda^{-\beta}\rho(1)$$

$$\overset{!}{=} t^{-\beta/\nu_\parallel}\rho(1) \qquad \underbrace{\rho(t) \sim t^{-\delta}}$$

$$\delta = \beta/\nu_\parallel$$

$t \to \infty$

($m \sim H^{1/\delta}$ at $T = T_c$, H = magnetic field, ISING model)

time as "external field"

$t \to +\infty$ to observe critical behaviour

Similarly

$P(t) \sim t^{-\delta'}$ ($p = p_c$) $\qquad \delta' = \beta'/\nu_\parallel$

for DP: $\beta = \beta' \to \delta = \delta' = \to \underbrace{\vartheta = \frac{d\nu_\perp}{\nu_\parallel} - 2\delta}$

Finite size scaling

$t \gg 1$; $\Delta \ll 1$; $V \gg 1$

($V = L$ in $d=1$) Volume)

$$\begin{cases} \rho(t, \Delta, V) \sim t^{-\beta/\nu_\parallel} f\Big(\underbrace{\Delta t^{1/\nu_\parallel}}_{\Delta \Lambda}, \underbrace{V t^{-d/z}}_{V \Lambda^{-d\nu_\perp}}\Big) \\ P(t, \Delta, V) \sim t^{-\beta'/\nu_\parallel} f\left(\Delta t^{1/\nu_\parallel}, V t^{-d/z}\right) \end{cases}$$

$$\underbrace{\Lambda = t^{1/\nu_\parallel}}$$