DRIVEN SYSTEMS

(phase transitions between phases with different currents)

TASEP model ($d=1$)

Total ASymmetric Exclusion Process

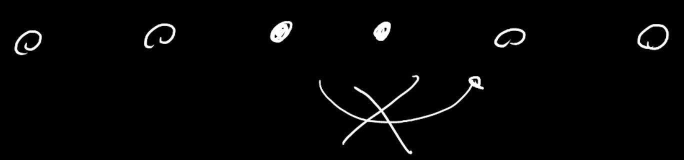
(single file diffusion) → "Traffic" model

Lattice gas: L sites with N particles

site K $(1 \leq K \leq L)$ → binary variable $n_K = \begin{cases} 0 & \text{empty ("hole")} \\ 1 & \text{occupied ("particle")} \end{cases}$

- Particles <u>always</u> move to the right (Totally asymmetric)

<u>only</u> if the nearby site is <u>empty</u> → <u>Exclusion</u>



[Lattice gas <u>in equilibrium</u>: same universality class as Ising model

→ no ph. tr. in $d=1$ (short-range interactions) (at $T>0$)

in $d=1$ out-of-eq. phase transition!]

Stochastic evolution rules (TASEP) in the "bulk"

- choose randomly a lattice site K $(1 \leq K \leq L)$

IF $n_k(t) = 1$ AND $n_{k+1}(0) = 0$

THEN $\rightarrow \begin{cases} n_k(t+1) = 0 \\ n_{k+1}(t+1) = 1 \end{cases}$ the particle moves to the right with probability 1

(ELSE)
Otherwise DO NOTHING

- PERIODIC BOUNDARY CONDITIONS (PBC)

$\Rightarrow$ all sites are "bulk" ones with $n_0 = n_{L-1}$

PBC break Detailed Balance $n_1 = n_{L+1}$

Stationarity condition: $\underbrace{\sum_{s'} W_{ss'} p^{ss}(s')} = \underbrace{\sum_{s'} W_{s's} p^{ss}(s)}$
$p^{ss}(s)$

we prove that $p^{ss}(s') \sim \text{constant}$ (all microstates have the same prob. at stationarity like $T = \infty$ at equilibrium)

$$\sum_{s'} W_{ss'} = \sum_{s'} W_{s's}$$

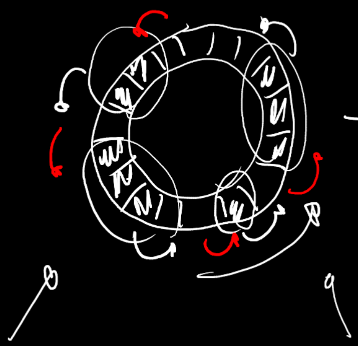
sum over incoming rates

sum over outgoing rates

# of ways in which S can be reached $\sim \underbrace{\sum_{s' \neq s} W_{ss'} = \sum_{s' \neq s} W_{s's}}$ $\sim$ # of ways in which S can be left

$\rightarrow$ both equal to

$m(s) = $ # of "blocks" $\rightarrow$ stretches of "consecutive" particles



$\rightarrow m(s) = 4$ possible moves to leave s

possible moves to reach s

$$W_{s's} = \begin{cases} 1/N & \text{for } s' \neq s \\ \dfrac{N-m(s)}{N} & \text{for } s' = s \\ 0 & \text{otherwise} \end{cases}$$

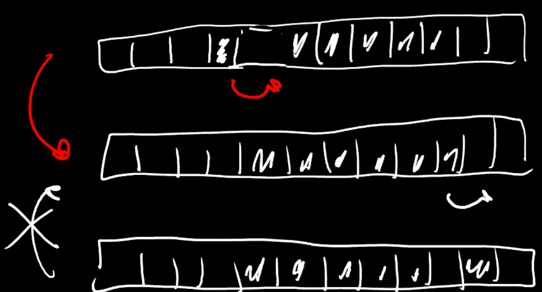
$m(s)$ possible states s' that can be reached from s

What about DB? if DB holds: $W_{ss'}\, p^{ss}(s') = W_{s's}\, p^{ss}(s)$ $\forall s, s'$

$\rightarrow \underbrace{W_{ss'} = W_{s's} \ \forall s, s'}$

$(p^{ss}(s) \sim \text{const.})$ & CANNOT BE TRUE

$$\boxed{\begin{array}{l} W_{ss'} \neq 0 \ (\text{I can reach } s \text{ from } s') \\ \Rightarrow W_{s's} = 0 \ (s' \text{ cannot be reached from } s) \end{array}}$$



s''

s

s'

$s'' \neq s'$

DB is BROKEN $\Rightarrow$ CURRENT

PBC $\rightarrow$ mean field provides the EXACT solution ($p(s) \sim$ const $\forall s$, no correlation between different states)

$p_i = \langle n_i \rangle$; $J_{i,i+1} = \langle n_i (1 - n_{i+1}) \rangle$ i $i+1$

(nonzero current only if $n_i = 1$ and $n_{i+1} = 0$)

$p^{ss}(s) \sim$ const $\Rightarrow$ $\underline{(PBC)}$ $\underline{\text{translation invariance}}$

$\rightarrow \underbrace{\rho_i = \rho = N/L}$ particle density (CONSERVED QUANTITY)

$\langle n_i \, n_{i+1} \rangle = \sum_s p^{ss}(s)\, n_i(s)\, n_{i+1}(s)$

ensemble average

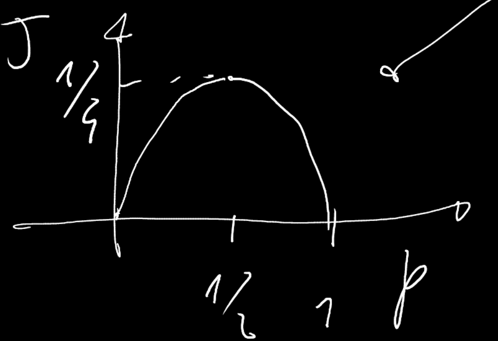
$= \dfrac{\#\text{ states with } i, i+1 \text{ occupied}}{\text{Total \# of states}}$

$= \binom{L-2}{N-2} \dfrac{1}{\binom{L}{N}} = \dfrac{N(N-1)}{L(L-1)} \underset{L,\, N \gg 1}{\sim} \left(\dfrac{N}{L}\right)^2 = \rho^2$

in the thermodyn. limit $\rightarrow$ $\underline{\text{No correlation}}$

$\langle n_i \, n_{i+1} \rangle = \langle n_i \rangle \langle n_{i+1} \rangle = \rho^2$

$J_{i,i+1} = J = \rho(1-\rho)$



$J_{max} = 1/4$ for $\rho = 1/2$

Symmetry for hole $\leftrightarrow$ particle

$\rho \longrightarrow 1-\rho$

TASEP OPEN Boundary conditions

$\rightarrow$ external driving in the system

N/L is NOT a conserved quantity

particles injected with rate α to the left ($k=1$)
" removed with rate β from the right ($k=L$)

$$\boxed{0 \le \alpha, \beta \le 1}$$

Formally: site 0 with fractional constant occupation $\rho_0 = \alpha$

site $L+1$ with fractional constant occupation $\rho_{L+1} = 1-\beta$

Stochastic evolution rules:

choose randomly $0 \le k \le L$

$$\begin{cases} \text{LEFT BOUNDARY}: k=0 \rightarrow n_1 = 0 \Rightarrow \begin{cases} n_1 = 1 & \text{with prob. } \alpha \\ n_1 = 0 & \text{with prob. } 1-\alpha \end{cases} \\ \text{BULK RULE}: 0 < k < L \quad n_k = 1 \text{ AND } n_{k+1} = 0 \text{ with prob. } 1 \\ \qquad \Rightarrow n_k = 0 \; ; n_{k+1} = 1 \\ \text{RIGHT BOUNDARY}: k = L \\ \qquad n_L = 1 \Rightarrow \begin{cases} n_L = 0 & \text{prob. } \beta \\ n_L = 1 & \text{prob. } 1-\beta \end{cases} \end{cases}$$

OPEN BOUNDARIES

$\rightarrow$ NO TRANSLATION INVARIANCE $\Rightarrow$ mean field is NOT EXACT

MF allows to recover the EXACT phase diagram in the (α, β) plane

general idea for MF approach: at stationarity

if $J_{i-1,i} \neq J_{i,i+1}$ but J needs to be uniform; ρ_i is NOT uniform

$$\frac{d\langle n_i\rangle}{dt} = J_{i-1,i} - J_{i,i+1} \neq 0 \text{ if } J_{i-1,i} \neq J_{i,i+1}$$

$$= \langle n_{i-1}(1-n_i)\rangle - \langle n_i(1-n_{i+1})\rangle$$

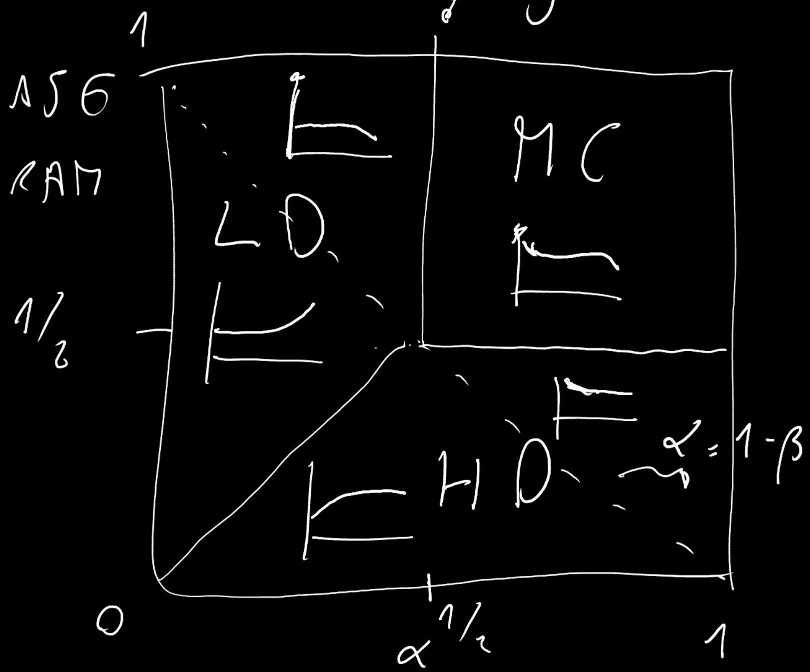
$$\text{MF} \simeq \rho_{i-1}(1-\rho_i) - \rho_i(1-\rho_{i+1})$$

At stationarity: $\underbrace{J_{i,i+1} = J_{i-1,i}}_{\downarrow} = J \quad \forall\, 0 \le i \le L$

$L+1$ equations for the $L+1$ unknowns ρ_i $(1 \le i \le L)$

$$\begin{cases} \alpha(1-\rho_1) = J \\ \rho_i(1-\rho_{i+1}) = J \\ \beta\rho_L = J \end{cases}$$

PHASE DIAGRAM

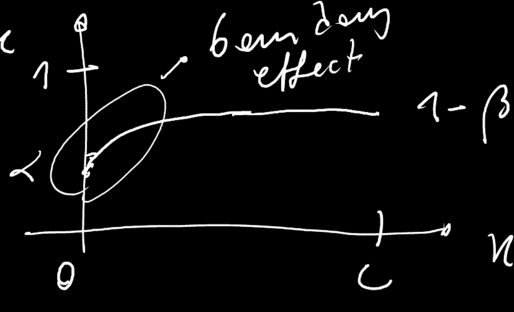


HD : high density phase

for $\beta < 1/2$; $\beta < \alpha$

→ { injected faster than removed
slow removal

→ particles "queuing" to the right



Stationary state

with $\rho_{bulk} = 1-\beta > 1/2$

$J = \beta(1-\beta)$ (non maximal)

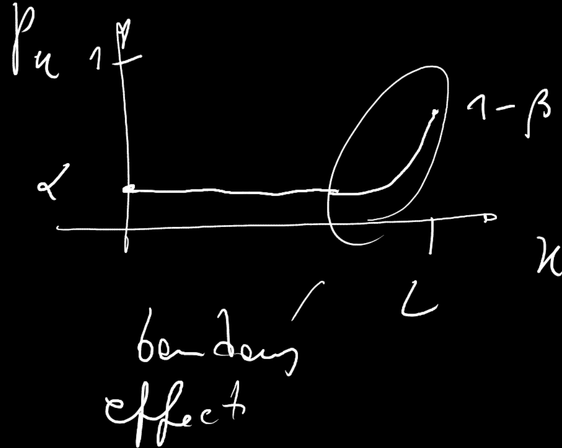
LD = Low density phase

(symmetry "particles" & "holes" $\alpha \leftrightarrow \beta$

for $\alpha < 1/2$ $\alpha < \beta$

removed faster than injected → "holes" queuing to the left

and slow injection



$$\rho_{bulk} = \alpha < 1/2$$

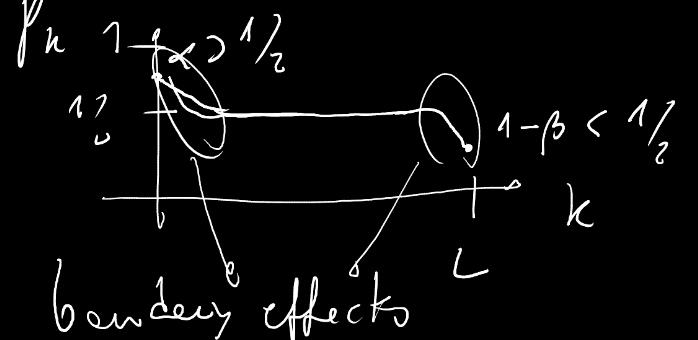
$$J = \alpha(1-\alpha)$$ not MAXIMAL

Maximal Current Phase

$\alpha \geq 1/2$; $\beta \geq 1/2$

(both removal/injection fast enough)

→ no kinetic bottlenecks in the system

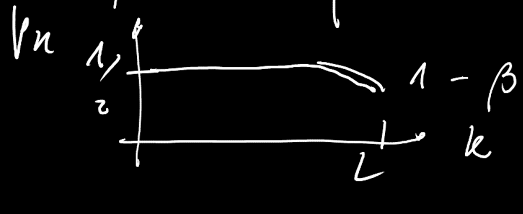


$$\rho_{bulk} = 1/2$$

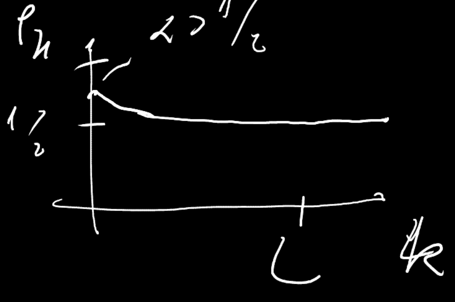
$$J_{bulk} = 1/4$$

PHASE TRANSITIONS

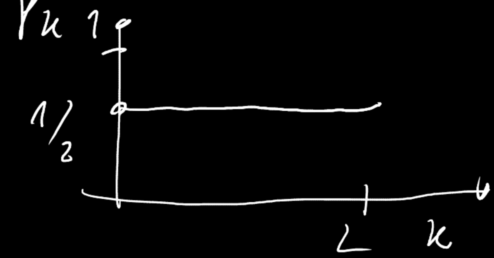
LD/MC $\beta > \alpha = 1/2$



HD/MC $\alpha > \beta = 1/2$



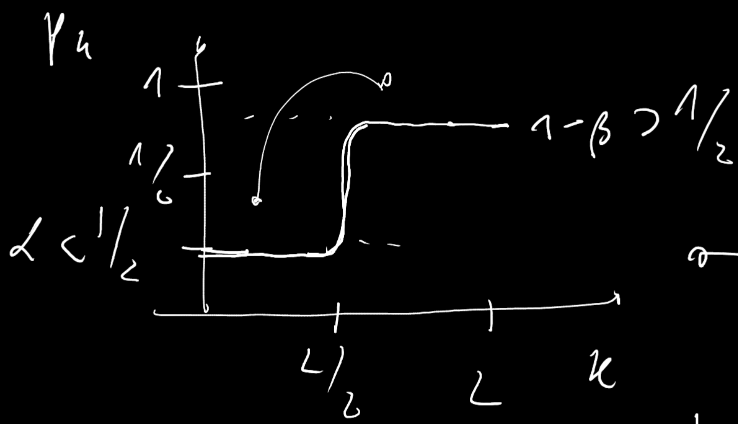
HD/LD/MC $\alpha = \beta = 1/2$



Continuous transitions

LD/HD $\alpha = \beta < 1/2$

discontinuous phase transition



→ MF coexisting different bulk behaviours

domain wall at $k = L/2$

EXACT solution: domain wall diffusing

NO SPONTANEOUS SYMMETRY BREAKING in the thermodynamic limit

(unlike what happens for the Ising model at $T < T_c$)

$\boxed{\text{BRIDGE MODEL}}$ → Spont. SB for non-equilibr. Ph. Tr.