Random Energy Model

$1 \le i \le 2^N$ energy levels with E_i random variable

i.i.d. $\quad \rho(E) = \frac{1}{\sqrt{\pi N}} \exp\left(-E^2/N\right)$

$$\varepsilon_i = E_i/N \to \rho(\varepsilon) = \sqrt{\frac{N}{\pi}} \exp\left(-N\varepsilon^2\right)$$

Microcanonical ensemble:

entropy density $\quad s(\varepsilon) = \begin{cases} \log 2 - \varepsilon^2 & |\varepsilon| < \varepsilon_* = \sqrt{\log 2} \\ -\infty & |\varepsilon| > \varepsilon_* \end{cases}$

Canonical ensemble:

partition function $\quad Z_N(\beta) \underset{N \gg 1}{\simeq} \int_{-\varepsilon_*}^{\varepsilon_*} d\varepsilon \exp\left[N\left(s(\varepsilon) - \beta\varepsilon\right)\right]$

SADDLE POINT $\quad Z_N(\beta) \simeq \exp\left[N\phi(\beta)\right]$

$$\phi(\beta) = \max_{-\varepsilon_* \le \varepsilon \le \varepsilon_*} \left(s(\varepsilon) - \beta\varepsilon\right)$$

Extremum condition for $s(\varepsilon) - \beta\varepsilon \to \frac{ds}{d\beta} = \beta$

$$\frac{ds}{d\beta} = -2\varepsilon$$

$d^2 s/d\beta^2 = -2 < 0$

$\to \beta = -2\varepsilon$ MAX. condition

- if $\beta/2 < \varepsilon_* \to \varepsilon = -\beta/2 \in [-\varepsilon_*, \varepsilon_*]$

$$\Rightarrow \phi(\beta) = \log 2 - \beta^2/4 + \beta^2/2 = \log 2 + \beta^2/4$$

• if $\beta/2 > \mathcal{E}_*$ $\to$ $\mathcal{E} = -\beta/2 \notin [-\mathcal{E}_*, \mathcal{E}_*]$

$\Rightarrow$ $\Phi(\beta) = +\beta\,\mathcal{E}_*$ $\leftarrow$ $S(\mathcal{E}) - \beta\mathcal{E}$ is MAX for $\mathcal{E} = -\mathcal{E}_*$

$(S(\mathcal{E}^*) = 0)$

$\Rightarrow$ PHASE TRANSITION AT $\beta_C = 2\,\mathcal{E}_* = 2\sqrt{\log 2}$

Thermodynamic potentials $(k_B = 1)$

free energy $$f(\beta) = -\frac{1}{\beta}\lim_{N\to\infty}\frac{\ln Z_N(\beta)}{N} = -\frac{\phi(\beta)}{\beta}$$

internal energy: $$u(\beta) = \frac{\partial}{\partial\beta}(\beta f) = -\frac{\partial\phi}{\partial\beta}$$

entropy: $$s(\beta) = \beta(u - f) = \beta^2\frac{\partial f}{\partial\beta}$$

• $\beta < \beta_C$ (high T phase)

$$f(\beta) = -\frac{\log 2}{\beta} - \beta/4 \; ; \quad u(\beta) = -\beta/2$$

$$s(\beta) = -\beta^2/4 + \log 2$$

exponentially large number $(N \to \infty)$ of configurations $E_i \approx -N\beta/2$

• $\beta > \beta_C$ (low T phase)

$$f(\beta) = -\mathcal{E}_* \; ; \quad u(\beta) = -\mathcal{E}_* \; ; \quad s(\beta) = 0$$

prototypical example for a "glassy" phase ("frozen")

relevant configurations with energy $E_i \approx -N\mathcal{E}_*$

Their number is NOT exponentially large ($S(-\varepsilon^*)=0$)

$\boxed{\text{MEASURE CONDENSATION}}$

↳ typical of glassy phases (glasses / spin glasses)

To quantify condensation:

participation ratio $\quad Y_N(\beta) = \sum_{j=1}^{2^N} \left[\mu_\beta(j)\right]^2$

$$\mu_\beta(j) = \frac{\exp(-\beta E_j)}{Z_N(\beta)} \quad \text{Boltz. prob.}$$

(in general $Y_N(\beta) = \sum_{j=1}^{2^N} p_j^2$ given $\sum_{j=1}^{2^N} p_j = 1$)

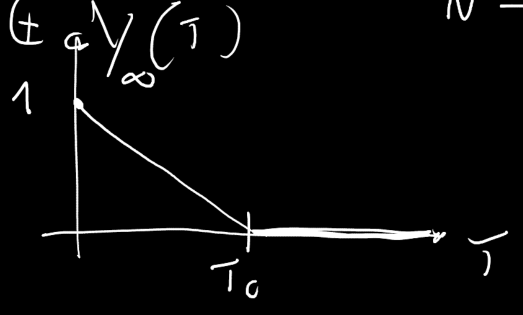
Limiting cases: $\quad - \; p_j = \delta_{j j_0} \to Y_N = 1$

$\boxed{0 \le Y_N \le 1}$ $\quad - \; p_j = \frac{1}{2^N} \;\forall j \to Y_N = \sum_{j=1}^{2^N} \frac{1}{(2^N)^2} = \frac{1}{2^N}$

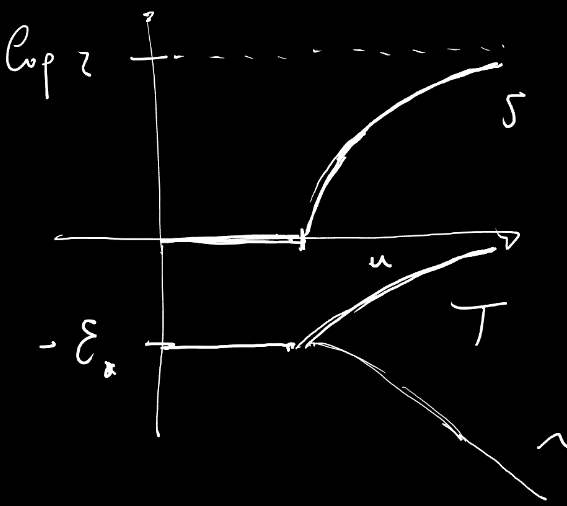
The higher Y_N, the more condensed the measure.

It can be proven that (REM)

$$\mathbb{E}\, Y_N(\beta) \xrightarrow[N\to\infty]{} \begin{cases} 0 & \beta < \beta_c \quad \text{high } T \text{ phase} \\ 1 - \beta_c/\beta & \beta > \beta_c \quad \text{low } T \text{ phase} \\ \text{''} \\ 1 - T/T_c & \end{cases}$$



$\beta = 1/T$; $\beta_c = 1/T_c$



2^{nd} order phase transition

$f(\beta)$ is continuous at $\beta = \beta_c$

$\underbrace{S(\beta),\ u(\beta)}$ are continuous at $\beta = \beta_c$

$dS/d\beta$; $du/d\beta$ are discont. at $\beta = \beta_c$

Quenched vs. annealed average

self-averaging quantities:

typical sample $\Rightarrow$ $\boxed{\text{quenched average}}$

$$f_{N,q} = \mathbb{E} f_N = -\frac{T}{N}\,\mathbb{E}\log Z_N(\beta)$$

$\rightarrow$ random variables FIXED (quenched) while averaging over Boltzmann distr.

<u>Annealed average</u>: random variables are allowed to thermalize according to Boltz. distr. together with other variables

$$f_{N,a} = -\frac{T}{N}\log\left(\mathbb{E} Z_N\right)$$ ← much easier to compute

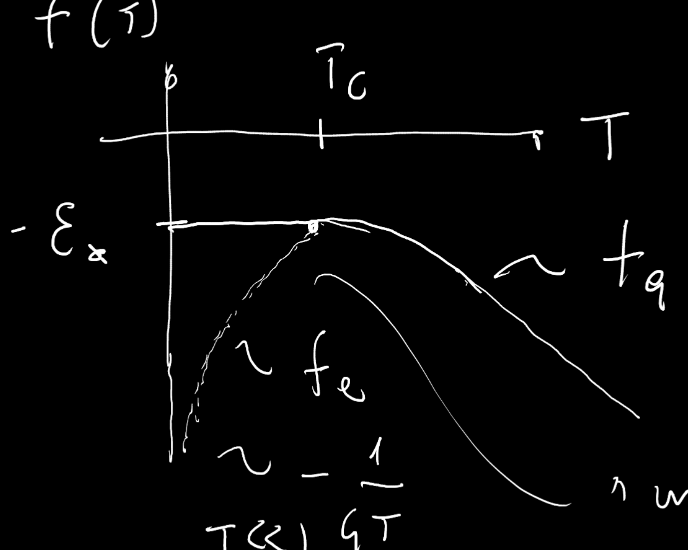
$f_{N,a}$ bound for $f_{N,q}$ → $\underbrace{f_{N,q}(T) \geq f_{N,a}(T)}$ ← Jensen's inequality for $-\log$ (convex)

<u>REM</u>: $$\mathbb{E} Z_N = \mathbb{E}\left[\sum_{i=1}^{2^N} \exp(-\beta E_i)\right] \overset{!}{=} 2^N\,\mathbb{E}\left[\exp(-\beta E)\right] = 2^N \exp(N\beta^2/4)$$

$\to f_{as,e}(\beta) = -\beta/4 - \log 2/\beta \quad \forall \beta$ | Gaussian PDF $P(\epsilon)$

(($=$ to $f(\beta)$ in the high energy phase)

$s_e(\beta) = -\beta^2/4 + \log 2 \; ; \; u_e(\beta) = -\beta/2$)



"unphysical" behaviour — $s_a(\beta) = -\frac{\beta^2}{4} + \log 2 < 0$

(negative entropy) $\beta > \beta_c$

$\boxed{\text{RANDOM CODE ENSEMBLE}}$

(proof of the direct statement of the channel coding theorem in the case of BSC)

$m \xrightarrow{\text{encoding}} \underline{x} \xrightarrow[Q(\underline{y}|\underline{x})]{\text{channel}} \underline{y} \xrightarrow{\text{decoding}} m'$

message in $\{0,1\}^M$

codeword in $\{0,1\}^N$

$\mathcal{Y}^N$

decoded message

$\mathcal{Y}$: output alphabet

$M < N \quad (R = M/N$ code rate ; $0 < R < 1)$

$\underline{x} = w(m)$ $\qquad Q(\underline{y}/\underline{x}) = \prod_{i=1}^N Q(y_i/x_i)$

memoryless channel

$m' = d(\underline{y})$

$\underline{y} \to \underline{x}^d(\underline{y}) \to m'$ in $\{0,1\}^M$

decoded codeword in $\{0,1\}^N$

difficult part of decoding

this part is easy if encoding $w(m)$ is injective

$m_1 \neq m_2 \Rightarrow \underline{x}(m_1) \neq \underline{x}(m_2)$

$\mathcal{C}_N$ = codebook = set of 2^M codewords in the space $\{0,1\}^N$ $(M < N)$

How many encoding maps $w(m) : \{0,1\}^M \to \{0,1\}^N$

for each of 2^M codewords we have to choose N bits

$2^{N 2^M}$ possible codes $\longrightarrow$ ensemble of possible codes

RCE = random code ensemble

each of the 2^{N2^M} possible codes is chosen with uniform probability

In practice: $\forall\, m \in \{0,1\}^M$

the bits of the corresponding codeword

$\underline{x}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \dots, x_N^{(m)})$

are obtained tossing a fair coin N times

the number of such pairs $m_1, m_2 \ll 2^M$ $(M \gg 1)$

(some encoding maps are NOT injective $\to$ $\underline{x}^{(m_1)} = \underline{x}^{(m_2)}$ for $m_1 \neq m_2$)

To understand when decoding is successful

→ GEOMETRY of RCE ($\sim$ microcanonical ensemble of REM)

set $\mathcal{C}_N$ of 2^M codewords in $\{0,1\}^N$ → Hamming space

(codebook)

(2^N corners of a unit side Hypercube in $\mathbb{R}^N$)

HAMMING DISTANCE

$d(\underline{x}, \underline{y})$ = # of different bits

$0 \leq d(\underline{x}, \underline{y}) \leq N$

Given a codeword $\underline{x}_0 \in \mathcal{C}_N$ $\quad d \in \mathbb{N}$

how many codewords are found at distance d from $\underline{x}_0$?

$\mathcal{N}_{\underline{x}_0}(d)$ = # of codewords $\underline{x} \in \mathcal{C}_N \,/\, d(\underline{x}, \underline{x}_0) = d$

distance enumerator ($\sim$ density of states in REM)

$\mathbb{E}\,\mathcal{N}_{\underline{x}_0}(d)$ does not depend on $\underline{x}_0$ for RCE

$\underline{x}_0 = 0\,0 \ldots 0$ (all bits are 0)

take $2^M - 1$ other points chosen randomly in Hamming space

$$\Rightarrow \mathbb{E}\,\mathcal{N}_{\underline{x}_0}(d) = \binom{N}{d} \left[\frac{2^{M-1}}{2^N} \right]$$

→ fraction of other codewords in Hamming space

(# of points in Hamming space at a distance d from $\underline{x}_0$)

Stirling formula: $N! \underset{N \gg 1}{\sim} \exp\left(N \log N - N\right)$

$\delta = d/N$

Thermodynamic limit

$N, d \to +\infty$

$\underbrace{\delta = d/N \text{ fixed}}_{\delta \in \mathbb{R}}$

$\underbrace{0 \leq \delta \leq 1}$

$$\frac{1}{N} \log_2 \binom{N}{d} \underset{N \gg 1}{\sim} \mathcal{H}_2(\delta)$$

$$\mathcal{H}_2(\delta) = -\delta \log_2 \delta - (1-\delta) \log_2 (1-\delta)$$

Shannon entropy of a Bernoulli process

RCE

$$\mathbb{E} \mathcal{N}_{x_0}(d) \underset{N \gg 1}{\simeq} 2^{N\left[R - 1 + \mathcal{H}_2(\delta)\right]}$$

growth rate $= R - 1 + \mathcal{H}_2(\delta)$ $\left(R = M/N\right)$

($\sim$ microcanonical entropy in REM)