Exact nonequilibrium quantum observable statistics: A large-deviation approach

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The exact statistics of an arbitrary quantum observable is analytically obtained. Due to the probabilistic nature of a sequence of intermediate measurements and stochastic fluctuations induced by the interaction with the environment, the measurement outcomes at the end of the system's evolution are random variables. Here, we provide the exact large-deviation form of their probability distribution, which is given by an exponentially decaying profile in the number of measurements. The most probable distribution of the measurement outcomes in a single realization of the system transformation is then derived, thus achieving predictions beyond the expectation value. The theoretical results are confirmed by numerical simulations of an experimentally reproducible two-level system with stochastic Hamiltonian.

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I. INTRODUCTION

Observables of a quantum system out of equilibrium return random outcomes fluctuating with a specific probability distribution [1]. By following the operational approach [2], a quantum stochastic process [3,4] can be envisaged as a nonequilibrium transformation modeled as the composition of stochastic evolutions characterized by semiclassical fluctuations of the system parameters [5–7] and intermediate quantum measurements [8,9]. In the former case, especially in quantum computing, stochastic fluctuations can arise from imperfections in the physical realization of the quantum system or through the interaction with the environment [10]. On the other hand, repeated quantum measurements could correspond to a process exchanging photons with the environment [11] or could be adopted to ensure the protection of coherent evolutions of a quantum system by decoherence (quantum Zeno dynamics) [12]. Experimentally, together with strong-coupling methods, such dynamics have been realized in several physical setups such as solid-state spins, superconducting qubits, and ultracold atomic gases [5,13–17]. They are also relevant in quantum metrology [18] to probe the phase evolution of an atomic ensemble by means of interleaved interrogations and feedback corrections [19,20]. In this scenario, by observing only once the dynamics of the system (single realization), the ensemble average of the measurement outcomes does not provide complete information about the statistics of the measured results. This becomes more evident when also one rare random event, i.e., a stochastic fluctuation with very small probability, occurs within the system dynamics [21–23]. Such a concept is at the heart of the large-deviation (LD) theory [24,25], dealing with the exponential decay of probabilities associated to large fluctuations in stochastic classical [26] and quantum systems [27–32].

In this paper, we derive a closed-form expression for the outcomes' statistics obtained by measuring a quantum system repeatedly monitored by an external observer and the dynamical evolution of which depends on random parameters. In particular, we prove that for a sufficiently large number m of intermediate quantum measurements the probability distribution of the last measurement results obeys the so-called LD principle [25,33,34]. This means that the behavior of the measurement outcomes' distribution is a decaying exponential in m, the exponent of which is equal to the relative Shannon entropy between the configurations of the stochastic system dynamics. In other words, only a rigorous description of the occurrence combinatorics of the parameters defining the stochastic evolution of the system allows for the full characterization of the outcomes' statistics, also *beyond* the Gaussian approximation given by the sole description of the measurement apparatus.

II. MODEL

Let us consider an arbitrary quantum system $\mathcal{S}$ within the Hilbert space $\mathcal{H}$. We assume that $\mathcal{S}$ is initialized in the quantum state with density matrix ρ_0 and that the Hamiltonian H of the system is time independent. At the level of the single trajectory, we assume that the random interaction between $\mathcal{S}$ and an environment $\mathcal{E}$ gives rise to a sequence of stochastic dynamical evolutions, separated by consecutive quantum projective measurements [35], following the postulates of quantum mechanics [36]. Hereafter, the index j denotes the dimension of $\mathcal{H}$, while the index α denotes the instants composing the temporal sequence of measurements. More specifically, we assume that the first $m-1$ measurements are performed on the quantum observable $\mathcal{O} \equiv \sum_j o_j \Pi_{o_j}$, where o_j are the outcomes of $\mathcal{O}$ and $\{\Pi_{o_j}\}$ is the set of projectors corresponding to the measured eigenvalues at time t_α. The mth measurement, instead, is performed on the quantum observable $\Theta \equiv \sum_j \theta_j \Pi_{\theta_j}$, the outcomes θ_j of which are recorded by the observer. According to the postulate of quantum

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measurement, the state ρ_α of $\mathcal{S}$ after the projective measurement at t_α is identically equal to one of the projectors defining the measurement observable. Then, between each projection event the system undergoes a dynamics, which is governed by the Hamiltonian H and described by the completely-positive and trace-preserving quantum map $\Phi(t_\alpha, t_0)[\rho_0] \equiv \Phi_\alpha[\rho_0]$ [37]. If we assume a quite large number of intermediate quantum measurements, the dynamics between measurements is described by a unitary operator, so that $\Phi_\alpha[\rho_{\alpha-1}]$ is simply given by the superoperator $\mathcal{U}_\alpha[\rho_{\alpha-1}] \equiv U_\alpha \rho_{\alpha-1} U_\alpha^\dagger$, where $U_\alpha \equiv \exp(-iH\tau_\alpha)$, $\hbar$ is set to unity, and $\tau_\alpha \equiv t_\alpha - t_{\alpha-1}$. We also assume that, in accordance with the recently introduced stochastic quantum Zeno phenomena [5,6,33], there exists for each propagator U_α at least one dynamical parameter λ, which is a fluctuating variable. For example, one could consider as in [38,39] that each λ_α is equal to the time interval τ_α, with τ_α random. Moreover, the λ's between the measurements are taken as constant and have a different random value only after the occurrence of a new measurement, according to the probability density function $p(\lambda)$. Here, $\vec{\lambda} \equiv (\lambda_1, \ldots, \lambda_m)$ denotes the sequence of dynamical parameters λ_α in a single realization of the system transformation. The fluctuating dynamical parameters λ_α, then, are taken as independent and identically distributed (i.i.d.) random variables sampled from $p(\lambda)$, since the environment is *a priori* unknown.

The stochastic nature of the measurement outcomes θ_m's at time t_m lies in the specific values assumed by $\vec{o}$ and $\vec{\lambda}$. Thus, the dynamics of $\mathcal{S}$ being stochastic, the single realization of the system density matrix $\rho_{m,\vec{o},\vec{\lambda}}$ at the end of its evolution is a fluctuating variable; i.e., given the sequences $\vec{o}$ and $\vec{\lambda}$, it is mapped into

$$\rho_{m,\vec{o},\vec{\lambda}} = \frac{\mathcal{P}_{\theta_m}\mathcal{U}_m\mathcal{P}_{o_{m-1}}\mathcal{U}_{m-1}\cdots\mathcal{P}_{o_1}\mathcal{U}_1[\rho_0]}{p_{\theta_m}(\vec{o},\vec{\lambda})}, \tag{1}$$

where $\mathcal{P}_{\mu_\alpha}[(\cdot)] \equiv \Pi_{\mu_\alpha}(\cdot)\Pi_{\mu_\alpha}$ is the measurement superoperator acting on $(\cdot)$ at t_α, with $\mu \in \{o, \theta\}$. Consequently,

$$p_{\theta_m}(\vec{o},\vec{\lambda}) \equiv \mathrm{Tr}\big[\mathcal{P}_{\theta_m}\mathcal{U}_m\mathcal{P}_{o_{m-1}}\mathcal{U}_{m-1}\cdots\mathcal{P}_{o_1}\mathcal{U}_1[\rho_0]\big], \tag{2}$$

i.e., $p_{\theta_m}(\vec{o},\vec{\lambda}) = \mathrm{Tr}[\Pi_{\theta_m}\rho_{m,\vec{o},\vec{\lambda}}]$, denotes the conditional probability to obtain the outcome θ_m from the measurement of Θ given the specific realization of the sequences $\vec{o}$ and $\vec{\lambda}$.

III. QUANTUM OBSERVABLE STATISTICS

In a single realization of the system transformation, $\rho_{m,\vec{o},\vec{\lambda}}$ and θ_m depend on $\vec{o}$ and $\vec{\lambda}$ and are random quantities. Thus, the following question naturally emerges: Which is the *best* description for the statistics of the measurement results $\theta_{m,j}$ from the observation of $\mathcal{S}$ at t_m? Three possible answers, characterized by an increasing degree of prediction accuracy, can be provided. First, one could describe the probabilistic expected result from the measured outcomes by using the expectation value $\mathrm{Tr}[\Theta\rho_m]$, with

$$\rho_m \equiv \langle\rho_{m,\vec{o},\vec{\lambda}}\rangle = \sum_{\vec{o}}\int_{\vec{\lambda}} d^m\vec{\lambda}\, p(\vec{\lambda})\rho_{m,\vec{o},\vec{\lambda}} \tag{3}$$

and $p(\vec{\lambda})$ denoting the occurrence joint probability of the dynamical parameters λ_α. It is worth noting that $\mathrm{Tr}[\Theta\rho_m]$ is equal to the ensemble average of all possible measurement outcomes. Second, the probability distribution of Θ's outcomes (green line in Fig. 1) can be introduced:

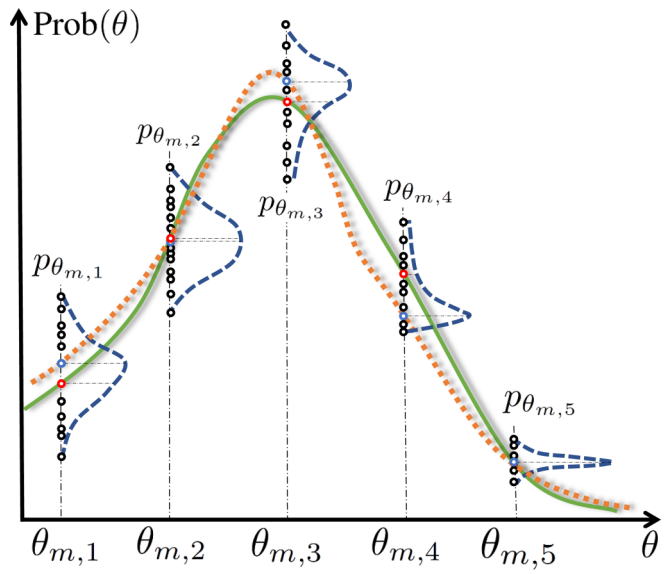

FIG. 1. Statistics of Θ's outcomes. By repeating several times the stochastic evolution of the system and measuring each of the outcomes $\theta_{m,j}$ at the final time instant t_m, an ensemble of conditional probabilities $p_{\theta_{m,j}}$ is obtained, where each of them is computed after the single realization of the system dynamics. Thus, just by counting the occurrence relative frequencies of the $p_{\theta_{m,j}}$'s, one can derive the corresponding probability distributions (blue dashed lines). If the number of realizations is relatively small, such distributions do not obey the Gaussian approximation. This means that one has to distinguish between two different statistics for θ's: one (green solid line) linking the ensemble averages $\langle p_{\theta_{m,j}}(\vec{o},\vec{\lambda})\rangle$ (red dots) of the conditional probabilities for each measurement outcome, and the other (orange dotted line)—also called *most probable distribution*—connecting all the realizations of $p_{\theta_{m,j}}$ in correspondence to the maximum value of the conditional probability distributions (blue dots).

$$\mathrm{Prob}(\theta) = \sum_j \delta(\theta - \theta_{m,j})\overline{p}_{\theta_{m,j}}, \tag{4}$$

where $\overline{p}_{\theta_{m,j}} \equiv \langle p_{\theta_{m,j}}(\vec{o},\vec{\lambda})\rangle = \mathrm{Tr}[\Pi_{\theta_{m,j}}\rho_m]$ and $\delta(\cdot)$ is the Kronecker delta. Equation (4) defines the statistics of Θ, while $\overline{p}_{\theta_{m,j}}$ (red dots in Fig. 1) is the probability to obtain on average the jth outcome $\theta_{m,j}$. Otherwise, the third option that we are here proposing is to adopt the probability distribution of the conditional probabilities $p_{\theta_{m,j}}(\vec{o},\vec{\lambda})$, i.e., $\mathrm{Prob}[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$ (in Fig. 1, the blue dashed lines for each outcome $\theta_{m,j}$), defined over all the possible realizations of the sequences $\vec{o}$ and $\vec{\lambda}$. Only in this way, by deriving $\mathrm{Prob}[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$, one will be able to get the most probable statistics (orange dotted line) for the quantum observable Θ connecting the blue dots in Fig. 1.

We start by observing that the $p_{\theta_{m,j}}(\vec{o},\vec{\lambda})$'s can be written as the product of the probabilities $|\langle\pi_{\mu_{\alpha-1}}|U_\alpha(\lambda_\alpha)|\pi_{\mu_\alpha}\rangle|^2$—also called dynamical transition probabilities—that the quantum state moves from $|\pi_{\mu_{\alpha-1}}\rangle$ to $|\pi_{\mu_\alpha}\rangle$ via the propagator $U_\alpha(\lambda_\alpha)$ (see the Appendix A for more details). Specifically, $|\pi_{\mu_\alpha}\rangle$'s are the eigenvectors that define the measurement projectors Π_{μ_α},

with μ equal to θ or o, depending on whether the observable Θ or $\mathcal{O}$ is measured. Thus, one has that

$$p_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \prod_{\alpha=1}^{m} \left|\langle \pi_{\mu_{\alpha-1}} | U_\alpha(\lambda_\alpha) | \pi_{\mu_\alpha}\rangle\right|^2, \quad \text{i.e.,} \tag{5}$$

$$\overline{p}_{\theta_{m,j}} = \sum_{\vec{o}} \prod_{\alpha=1}^{m} \int_{\lambda_\alpha} d\lambda_\alpha p(\lambda_\alpha) q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha), \tag{6}$$

where $q_\alpha \equiv |\langle \pi_{\mu_{\alpha-1}} | U_\alpha(\lambda_\alpha) | \pi_{\mu_\alpha}\rangle|^2$; $\pi_{\mu_0} \equiv \pi_0$ (with $\rho_0 = |\pi_0\rangle\langle\pi_0|$); $\mu_\alpha \equiv o_\alpha$ for $\alpha = 1,\ldots,m-1$; and $\pi_{\mu_m} \equiv \pi_{\theta_{m,j}}$.

IV. LARGE-DEVIATION FORMALISM

We now derive the main result of this paper, namely, a closed-form expression of the $p_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ distributions. Notice that if we just characterize the quantum observable Θ at t_m and at the same time the hypotheses of the central limit theorem are respected then the statistics of its outcomes is well represented by a Gaussian probability distribution. In fact, with this assumption, outliers in the outcomes statistics are simply classified as the result of a nonmodeled experimental noise on the measurement apparatus. Thus, by increasing the number of realizations, the occurrence probability of the outliers naturally decreases and the Gaussian distribution well fits the data. However, this evidence is no longer valid if we assume that the statistics of $\theta_{m,j}$'s is given by an arbitrary stochastic transformation governing the dynamics of the system. In such a case, the configuration space defined by the occurrence of random events during its evolution becomes exponentially larger, but allows for the description of the outliers' statistics with occurrence probability greater than zero and not satisfying the Gaussian approximations. Therefore, we expect to increase our prediction power on the outcomes' distribution by statistically characterizing the trajectories of the system before the measurement of Θ. This amounts to computing the statistics of the dynamical transition probabilities $q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha)$ as defined in Eq. (5). Briefly, the procedure to derive the exact LD form of $\text{Prob}[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$ is to take the logarithm of the conditional probabilities $p_{\theta_{m,j}}(\vec{o},\vec{\lambda})$, i.e., $l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) \equiv \ln p_{\theta_{j,m}}(\vec{o},\vec{\lambda}) = \sum_{\alpha=1}^{m} \ln q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha)$, compute its distribution, and then apply the contraction principle from LD theory [25].

For the sake of clarity, let us here consider that the measurement bases of $\mathcal{O}$ and Θ, with $[\mathcal{O},\Theta] \neq 0$, belong to a set of finite dimension, i.e., that each measurement eigenvector $|\pi_{\mu_\alpha}\rangle$ admits only a finite number d_π of elements $|\pi^{(j)}\rangle$. Then, the distribution of $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ is obtained as following. By following a common procedure in LD theory, the terms of $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ are grouped as a function of the number of times each dynamical transition probability $q_{j_\text{B},j_\text{A}}(\lambda) \equiv |\langle \pi^{(j_\text{B})} | U(\lambda) | \pi^{(j_\text{A})}\rangle|^2$ occurs, where the superscripts B and A stand, respectively, for "before" and "after" the evolution via the propagator $U(\lambda)$. In this way, $l_{\theta_{m,j}}$ is recast in the following sum of i.i.d. random variables:

$$l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \sum_{j_\text{B},j_\text{A}=1}^{d_\pi} \int_\lambda n_{j_\text{B},j_\text{A}}(\lambda) \ln q_{j_\text{B},j_\text{A}}(\lambda) d\lambda, \tag{7}$$

with $n_{j_\text{B},j_\text{A}}(\lambda)$ denoting the relative frequencies for the occurrence of the $q_{j_\text{B},j_\text{A}}(\lambda)$'s, usually different from the corresponding probability value $p_{j_\text{B},j_\text{A}}(\lambda)$. The deviation between them vanishes only by attaining a complete statistics for each admissible value of $q_{j_\text{B},j_\text{A}}(\lambda)$. This latter condition can be in principle realized, but likely a quite long system dynamical evolution could be required.

The probability distribution of a sum of n i.i.d. random terms can be always written as an exponential, linearly decaying in n with n large. In particular, regarding $\text{Prob}(l_{\theta_{m,j}})$, it is given by an exponential distribution decaying in the number m of projective measurements (see the Appendix B for more details), i.e.,

$$\text{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] \asymp \exp\big\{-m\,I\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big]\big\}. \tag{8}$$

In Eq. (8), the function $I(l_{\theta_{m,j}}/m)$, also called the *rate function* associated to the probability distribution $\text{Prob}(l_{\theta_{m,j}})$, equals to

$$I\big(l_{\theta_{m,j}}/m\big) \equiv \sum_{j_\text{B},j_\text{A}=1}^{d_\pi} \int_\lambda f_{j_\text{B},j_\text{A}}(\lambda) \ln\left(\frac{f_{j_\text{B},j_\text{A}}(\lambda)}{p_{j_\text{B},j_\text{A}}(\lambda)}\right) d\lambda, \tag{9}$$

where $f_{j_\text{B},j_\text{A}}(\lambda) \equiv n_{j_\text{B},j_\text{A}}(\lambda)/m$ for each set $(j_\text{B},j_\text{A},\lambda)$ of system parameters. The rate function $I(l_{\theta_{m,j}}/m)$ is the *Kullback-Leibler distance* (or relative entropy) between the set $\{f_{j_\text{B},j_\text{A}}(\lambda)\}$ of *scaled* relative frequencies and the set of probabilities $\{p_{j_\text{B},j_\text{A}}(\lambda)\}$ and, thus, has the properties to be positive and convex. Equation (8) is valid in the limit of m large and indicates a *unique nonequilibrium weighted partition* of the system configuration space (see the Appendix B for the proof). For a small value of m, indeed, the distribution $\text{Prob}(l_{\theta_{m,j}})$ cannot be uniquely determined. The latter can be considered as a universal property of any dynamical evolutions given by the composition of quantum maps and projections. The alternative LD expression of Eq. (8), providing the formal definition of $I(l_{\theta_{m,j}}/m)$, is

$$\lim_{m\to\infty} -\frac{1}{m} \ln \text{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] = I\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big], \tag{10}$$

where the number m of measurements is assumed ideally infinite. If Eq. (10), also called the *large-deviation approximation*, holds, it means that the dominant behavior of $\text{Prob}(l_{\theta_{m,j}})$ is *convergent* and identically equal to a decaying exponential in m.

As last, through the contraction principle [40], the distribution $\text{Prob}[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$ is obtained:

$$\text{Prob}\big[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] = \int \text{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big]\delta\big(l_{\theta_{m,j}} - \ln p_{\theta_{m,j}}\big) d\, l_{\theta_{m,j}}. \tag{11}$$

Then, by applying the saddle-point method [41], one has that

$$\text{Prob}\big[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] \asymp \exp\big\{-m\,J\big[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big]\big\}, \tag{12}$$

where

$$J\big[p_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big] \equiv \min_{l_{\theta_j}:\, l_{\theta_j} = \ln p_{\theta_j}} I\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big]. \tag{13}$$

The result is that a quantum system exhibiting stochastic evolutions (e.g., due to random interactions with the environment) and repeatedly monitored by an observer tends to

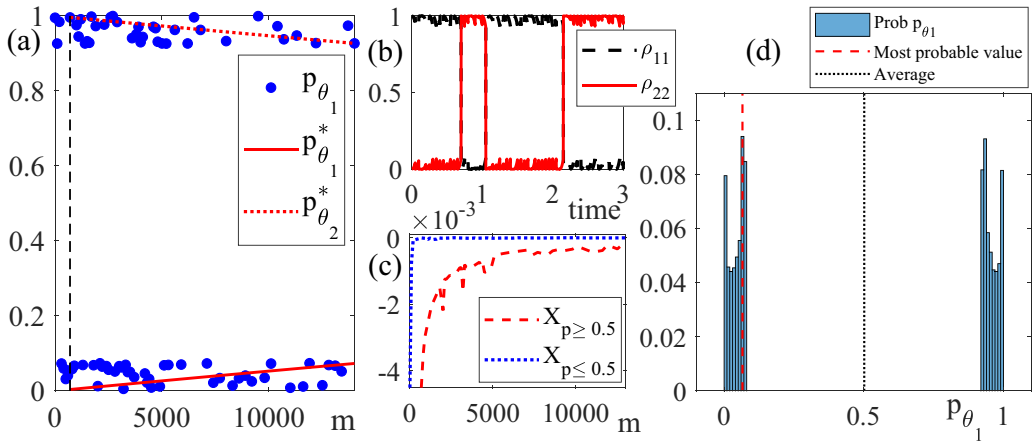

FIG. 2. Experimental proposal: Exact quantum observable statistics of a 2LS with stochastic Hamiltonian. (a) Theoretical most probable values $p^\star_{\theta_1}$ and $p^\star_{\theta_2}$ as a function of m compared with different values of the probability p_{θ_1}, which has been numerically obtained from single realizations of the system transformation. (b) Time behavior of system populations $\rho^{(11)}$ and $\rho^{(22)}$ by numerically solving the stochastic evolution of the 2LS for one realization of $\vec{o}$ and $\vec{\varphi}$. (c) Plot of $\ln(p_{\theta_1})/m$ from data of inset (a). (d) Histogram of p_{θ_1}, with $m = 14 \times 10^3$, over 20×10^3 repetitions of the system dynamics.

reach in probability a unique configuration defined by specific probability distributions of its characteristic parameters.

Most probable distribution

The expression of the most probable distribution of Θ's outcomes is here discussed. In general, from the knowledge of the rate function $I(\xi/m)$ associated to Prob(ξ), we can then compute the most probable value $\xi^\star$, representing the best prediction for the random variable ξ in a single realization of the system dynamics. Specifically, the most probable value of the log-conditional probability $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$, i.e., $l^\star_{\theta_{m,j}}$, is obtained by evaluating the value at which the rate function $I(l_{\theta_{m,j}}/m)$ is minimized as a function of $l_{\theta_{m,j}}$. As proved in the Appendix C, the closed-form expression of $l^\star_{\theta_{m,j}}$ is

$$l^\star_{\theta_{m,j}} = m \sum_{j_B,j_A=1}^{d_\pi} \int_\lambda p_{j_B,j_A}(\lambda) \ln q_{j_B,j_A}(\lambda)\, d\lambda, \tag{14}$$

corresponding to the statement that $n_{j_B,j_A}(\lambda) = m\, p_{j_B,j_A}(\lambda)$ for every (j_B, j_A, λ). This means that for large m the most probable trajectories of the system dynamics are those allowing for the equality between the scaled relative frequencies $f_{j_B,j_A}(\lambda)$ and the occurrence probabilities $p_{j_B,j_A}(\lambda)$. Only this condition minimizes the relative Shannon entropy between the configurations induced by the stochastic dynamics of the system. Once more, it is worth observing the importance of imposing $m \to \infty$; indeed, the value of the scaled relative frequencies f_{j_B,j_A} can get closer to that of the probabilities p_{j_B,j_A} only if the number of configurations generated by the stochasticity within the dynamics is as large as possible. Finally, to derive the most probable value of the conditional probabilities $p^\star_{\theta_{m,j}}$, we use again the contraction principle from LD theory, but this time on the functional relation between $l_{\theta_{m,j}}$ and $p_{\theta_{m,j}}$, i.e., $p_{\theta_{m,j}} = e^{l_{\theta_{m,j}}}$:

$$p^\star_{\theta_{m,j}} = \exp\left(m \sum_{j_B,j_A=1}^{d_\pi} \int_\lambda p_{j_B,j_A}(\lambda) \ln q_{j_B,j_A}(\lambda)\, d\lambda \right). \tag{15}$$

V. DISCUSSION

Having proved that the nonequilibrium statistics of an arbitrary observable obeys the LD principle for a sufficiently large number of intermediate projection events, the answer to the original question is given by the distribution of Θ's outcomes with occurrence conditional probabilities equal to $p^\star_{\theta_{m,j}}$. Such a distribution is denoted as the *most probable distribution* [see, e.g., Fig. 2(a) inset]:

$$\mathrm{Prob}^\star(\theta) \equiv \frac{1}{\mathcal{N}} \sum_j \delta(\theta - \theta_{m,j}) p^\star_{\theta_{m,j}}, \tag{16}$$

which has to be normalized by the factor $\mathcal{N}$ so as to ensure that $\int \mathrm{Prob}^\star(\theta) d\theta = 1$. Accordingly, the predictions about the result of single-shot measurements from Θ according to the most probable distribution $\mathrm{Prob}^\star(\theta)$ are expected to be more accurate than the ones that we would obtain by directly computing the expectation value of the measurement outcomes. This is because *large fluctuations* within the evolution of the system are now properly weighted and thus correctly included in the outcome distributions. In this regard, also quantum noise sensing techniques [42] could be used, so as to improve the *a priori* information on $p(\lambda)$.

In conclusion, the average $\overline{p}_{\theta_{m,j}}$ being explicitly equal to

$$\langle p_{\theta_{m,j}}(\vec{o},\vec{\lambda})\rangle = \left(\sum_{j_B,j_A=1}^{d_\pi} \int_\lambda p_{j_B,j_A}(\lambda) q_{j_B,j_A}(\lambda) d\lambda \right)^m, \tag{17}$$

one can observe a deviation between the most probable values $p^\star_{\theta_{m,j}}$ and the probabilities $\overline{p}_{\theta_{m,j}}$ to measure on average the outcome $\theta_{m,j}$ at the final time instant t_m. Specifically, by applying Jensen's inequality to $p^\star_{\theta_{m,j}}$ and $\overline{p}_{\theta_{m,j}}$, the inequality $p^\star_{\theta_{m,j}} \leqslant \overline{p}_{\theta_{m,j}}$ is obtained. The latter well justifies the introduction of the normalization factor $\mathcal{N}$ in Eq. (16). In fact, the normalization of $\mathrm{Prob}^\star(\theta)$ has to be ensured, and the values of the $p^\star_{\theta_{m,j}}$'s are thus corrected by dividing for $\mathcal{N} \equiv \sum_j p^\star_{\theta_{m,j}} \leqslant 1$, so that $p^\star_{\theta_{m,j}}/\mathcal{N} = 1 - (\sum_{k,\, k\neq j} p^\star_{\theta_{m,k}})/\mathcal{N}, \forall\, j = 1, \ldots, \dim(\mathcal{H})$.

VI. EXPERIMENTAL PROPOSAL

Let us consider a two-level system (2LS) with Hamiltonian $H = \frac{1}{2}\omega(\cos\varphi\sigma_z - \sin\varphi\sigma_x)$, with $\omega = 11$ (in natural units so that $\hbar = 1$) and σ_x and σ_z Pauli matrices. This system, realized, e.g., within a Bose-Einstein condensate of rubidium atoms confined by a magnetic microtrap as in [5], is intrinsically stochastic due to random fluctuations of the angle φ. They can be produced, for example, by a random coupling between the system and the laser source that coherently drives it. In particular, we assume sampling of φ from the Rayleigh probability distribution $p(\varphi) = (\varphi/\sigma^2)\exp[-\varphi^2/(2\sigma^2)]$ with scale parameter $\sigma = 1$. Then, the 2LS, starting from the ground $|0\rangle$, is monitored by a sequence of m repeated projective measurements. They are separated by the constant time interval $\tau = 0.05$ (in natural units) and defined by the projectors $\Pi_1 \equiv |0\rangle\langle 0|$ and $\Pi_2 \equiv |1\rangle\langle 1|$. After each dynamical evolution with whole duration $t_m = m\tau$, the quantum observable

$$\Theta \equiv \sum_{j=1}^{2} \theta_j \Pi_{\theta_j} = \rho^{(11)}\Pi_1 + \rho^{(22)}\Pi_2 \tag{18}$$

is measured, where the measurement outcomes $\theta_1 \equiv \rho^{(11)}$ and $\theta_2 \equiv \rho^{(22)} = 1 - \rho^{(11)}$ identify the percentage of system population, respectively, in the ground and excited states: $|0\rangle$ and $|1\rangle$, respectively. Notice that, $\mathcal{S}$ being a 2LS, we analyze the fluctuations of only the conditional probability $p_{\theta_1}(\vec{o}, \vec{\varphi}) \equiv \mathrm{Tr}[\Pi_{\theta_1}\rho_{m,\vec{o},\vec{\varphi}}]$.

In Fig. 2(a) we show the most probable values of p_{θ_1}, i.e., $p^{\star}_{\theta_1}$ (solid red curve), which has been computed by plotting the analytical expression of Eq. (15) as a function of m. The following procedure has been followed. First, we have numerically computed the probabilities p_{θ_1} [blue points in Fig. 2(a)] to measure the measurement outcome θ_1 after each realization of the system stochastic transformation [see Fig. 2(b) inset]. Then, we have derived the quantity $\ln(p_{\theta_1})/m$, by distinguishing between values of p_{θ_1} smaller [dotted blue curve in Fig. 2(c) inset] and greater [dashed red curve in Fig. 2(c) inset] than $1/2$. As it can be observed, only the dotted blue curve is practically constant for $m \geqslant 700$ and equal to $X \simeq -4.22 \times 10^{-6}$. In particular, from the theoretical findings, we can state that for m large $\ln(p^{\star}_{\theta_1}) \asymp mX$ [solid red curve in Fig. 2(a)] with

$$X \equiv \sum_{j_{\mathrm{B}}, j_{\mathrm{A}}} \int p_{j_{\mathrm{B}}, j_{\mathrm{A}}}(\varphi) \ln\left[q_{j_{\mathrm{B}}, j_{\mathrm{A}}}(\varphi)\right] d\varphi, \quad \text{i.e.,} \tag{19}$$

$$X = \frac{1}{N_{\mathrm{meas}}} \int d\varphi p(\varphi) \sum_{k,j=0}^{1} \ln |\langle k|U(\varphi)|j\rangle|^2 \tag{20}$$

for this specific example, where $U(\varphi) \equiv \exp[-iH(\varphi)\tau]$ and N_{meas} encodes the occurrence probabilities of the intermediate quantum measurement projectors. As a further test, we have numerically obtained the probability distribution of p_{θ_1} [Fig. 2(d) inset] with $m = 14 \times 10^3$, by repeating 20×10^3 times the dynamics of the system, and computed the value of $p^{\mathrm{peak}}_{\theta_1}$ at which $\mathrm{Prob}(p_{\theta_1})$ admits the peak; we measured $p^{\mathrm{peak}}_{\theta_1} = 0.0736$ and an average value $\overline{p}_{\theta_1} = 0.5016$. This value should equal to the most probable value $p^{\star}_{\theta_1}$ and this has been effectively verified by the numerics, since from the solid red curve in Fig. 2(a) we find for $m = 14 \times 10^3$ the value $p^{\star}_{\theta_1} = 0.0731$.

VII. CONCLUSIONS

These results combine out-of-equilibrium quantum systems with statistical mechanics methods: They are expected to be a tool to defeat noise in a quantum system, since they provide predictions of the stochastic evolution of the system after the single realization.

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APPENDIX A: MEASUREMENT CONDITIONAL PROBABILITIES

Let us express the measurement projectors Π_{μ_α} applied at time instants t_α and $\alpha = 1, \ldots, m$ as a function of their eigenvectors $|\pi_{\mu_\alpha}\rangle$, i.e., $\Pi_{\mu_\alpha} \equiv |\pi_{\mu_\alpha}\rangle\langle\pi_{\mu_\alpha}|$, with $\mu \in \{\theta, o\}$. Thus, by substituting the definition of the measurement superoperators $\mathcal{P}_{\theta_m}$ and $\mathcal{P}_{o_\alpha}$ into the equation

$$p_{\theta_m}(\vec{o}, \vec{\lambda}) \equiv \mathrm{Tr}\big[\mathcal{P}_{\theta_m}\mathcal{U}_m\mathcal{P}_{o_{m-1}}\mathcal{U}_{m-1}\cdots\mathcal{P}_{o_1}\mathcal{U}_1[\rho_0]\big], \tag{A1}$$

the conditional probability $p_{\theta_{m,j}}(\vec{o}, \vec{\lambda})$ can be written as the product of the transition probabilities (random terms) that the quantum state moves from $|\pi_{\mu_{\alpha-1}}\rangle$ to $|\pi_{\mu_\alpha}\rangle$ via the unitary operator $U_\alpha(\lambda_\alpha)$:

$$\begin{aligned} p_{\theta_{m,j}}(\vec{o}, \vec{\lambda}) &= \mathrm{Tr}\big[\Pi_{\theta_{m,j}}U_m\Pi_{o_m-1}U_{m-1}\cdots\Pi_{o_1}U_1\rho_0U_1^{\dagger}\Pi_{o_1}\cdots U_{m-1}^{\dagger}\Pi_{o_{m-1}}U_m^{\dagger}\Pi_{\theta_{m,j}}\big] \\ &= \mathrm{Tr}\big[|\pi_{\theta_{m,j}}\rangle\langle\pi_{\theta_{m,j}}|U_m|\pi_{o_{m-1}}\rangle\langle\pi_{o_{m-1}}|U_{m-1}\cdots|\pi_{o_1}\rangle\langle\pi_{o_1}|U_1\rho_0U_1^{\dagger}|\pi_{o_1}\rangle\langle\pi_{o_1}|\cdots U_{m-1}^{\dagger}|\pi_{o_{m-1}}\rangle\langle\pi_{o_{m-1}}|U_m^{\dagger}|\pi_{\theta_{m,j}}\rangle\langle\pi_{\theta_{m,j}}|\big] \\ &= \langle\pi_{o_1}|U_1\rho_0U_1^{\dagger}|\pi_{o_1}\rangle \cdot \prod_{k=2}^{m-1} |\langle\pi_{o_{k-1}}|U_k|\pi_{o_k}\rangle|^2 \cdot |\langle\pi_{o_{m-1}}|U_m|\pi_{\theta_{m,j}}\rangle|^2. \end{aligned} \tag{A2}$$

ρ_0 is the initial density matrix of the open quantum system $\mathcal{S}$ before the transformation induced by random interactions with the environment $\mathcal{E}$ and the monitoring by an observer. Moreover, we also assume that ρ_0 is defined by the pure state $|\pi_0\rangle$, i.e., $\rho_0 \equiv |\pi_0\rangle\langle\pi_0|$. For example, this assumption is verified when the measurement outcomes at the final time t_m are obtained by a two-time measurement scheme [8]. In such a case, indeed, ρ_0 is the density matrix of $\mathcal{S}$ after the first measurement of the scheme, and thus it is described only by a pure state. Therefore, under this further hypothesis, one has that $\langle\pi_{o_1}|U_1\rho_0 U_1^\dagger|\pi_{0_1}\rangle = |\langle\pi_0|U_1|\pi_{0_1}\rangle|^2$, so that

$$p_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \prod_{\alpha=1}^{m} \left|\left\langle\pi_{\mu_{\alpha-1}}\middle|U_\alpha(\lambda_\alpha)\middle|\pi_{\mu_\alpha}\right\rangle\right|^2, \tag{A3}$$

with $\pi_{\mu_0} \equiv \pi_0$ and $\mu_\alpha \equiv o_\alpha$ for $\alpha = 1,\ldots,m-1$ and $\pi_{\mu_m} \equiv \pi_{\theta_{m,j}}$.

APPENDIX B: DERIVATION OF Prob$[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$ VIA LD THEORY

The log probability $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ is the logarithm of the conditional probability $p_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ defining the probability to obtain the jth measurement outcome $\theta_{m,j}$ at the time instant t_m, conditioned to the specific realizations of the sequences $\vec{o}$ and $\vec{\lambda}$. More formally,

$$\begin{aligned} l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) &\equiv \ln p_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \sum_{\alpha=1}^{m} \ln q_\alpha(\mu_{\alpha-1},\mu_\alpha,\lambda_\alpha) \\ &= \sum_{\alpha=1}^{m} \ln\left(\left|\left\langle\pi_{\mu_{\alpha-1}}\middle|U_\alpha(\lambda_\alpha)|\pi_{\mu_\alpha}\right\rangle\right|^2\right). \end{aligned} \tag{B1}$$

For the sake of clarity, let us assume the following two hypotheses.

(i) $p(\lambda)$ is assumed to be a d_λ-dimensional Bernoulli distribution, with the result that at each time instant t_α the parameter λ_α takes on d_λ possible values $\lambda^{(1)},\ldots,\lambda^{(d_\lambda)}$ with corresponding probabilities $p_\lambda^{(1)},\ldots,p_\lambda^{(d_\lambda)}$ so that $\sum_{i=1}^{d_\lambda} p_\lambda^{(i)} = 1$. The index i denotes the values that can be assumed by λ.

(ii) The measurement bases of Θ and $\mathcal{O}$, given by the set of eigenvectors $\{|\pi_{\mu_\alpha}\rangle\}$, belong to a set of finite dimension and this means that each ket $|\pi_{\mu_\alpha}\rangle$ admits only a finite number d_π of elements $|\pi^{(j)}\rangle$ with probability p_π. Notice that the index for the dimensionality of the measurement basis configurations, as well as of $\mathcal{H}$, is j.

By following a common procedure in LD theory, we group the terms of $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ as a function of the number of times (relative frequencies) each dynamical transition probability $q_{j_B,i,j_A} \equiv |\langle\pi^{(j_B)}|U(\lambda^{(i)})|\pi^{(j_A)}\rangle|^2$ occurs during the transformation of the system. We denote with n_{j_B,i,j_A} the relative frequencies for the occurrence of the q_{j_B,i,j_A}'s. The corresponding occurrence probabilities, instead, are denoted as p_{j_B,i,j_A}. Over many realizations of the system transformation, one can reasonably assume that p_π is sampled by a uniform distribution, such that $p_\pi = 1/d_\pi$ and $p_{j_B,i,j_A} \equiv p_\pi^{(j_B)} p_\lambda^{(i)} p_\pi^{(j_A)} = p_\lambda^{(i)}/d_\pi^2$. The latter procedure directly leads to the analytical expression of $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$, i.e.,

$$l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \sum_{j_B=1}^{d_\pi}\sum_{i=1}^{d_\lambda}\sum_{j_A=1}^{d_\pi} n_{j_B,i,j_A} \ln q_{j_B,i,j_A}. \tag{B2}$$

Equation (B2) shows us that $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ can be written as the sum of the i.i.d. dynamical transition probabilities q_{j_B,i,j_A}, weighted by the relative frequencies n_{j_B,i,j_A} corresponding to the occurrence statistics of the q_{j_B,i,j_A}'s. Notice that the relative frequencies n_{j_B,i,j_A} are usually different from the corresponding probability values p_{j_B,i,j_A}. Then, by taking Eq. (B2), the distribution probability of $l_{\theta_{m,j}}(\vec{o},\vec{\lambda})$ is given by

$$\begin{aligned} &\mathrm{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] \\ &\quad= \frac{m!}{\prod_{j_B,i,j_A} n_{j_B,i,j_A}!} \prod_{j_B,i,j_A} \left(p_{j_B,i,j_A}\right)^{n_{j_B,i,j_A}} \\ &\qquad\times \delta\left[l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) - \sum_{j_B,i,j_A} n_{j_B,i,j_A} \ln q_{j_B,i,j_A}\right], \end{aligned} \tag{B3}$$

where $\delta(\cdot)$ denotes the Kronecker delta. Let us observe that in Eq. (B3), to simplify the notation, we have used the symbols $\sum_{j_B,i,j_A}$ and $\prod_{j_B,i,j_A}$ to denote, respectively, $\sum_{j_B=1}^{d_\pi}\sum_{i=1}^{d_\lambda}\sum_{j_A=1}^{d_\pi}$ and $\prod_{j_B=1}^{d_\pi}\prod_{i=1}^{d_\lambda}\prod_{j_A=1}^{d_\pi}$. Then, by imposing in Eq. (B3) the condition given by the Kronecker delta, one has that

$$\mathrm{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] = \frac{m!}{\prod_{j_B,i,j_A} \widehat{n}_{j_B,i,j_A}!} \prod_{j_B,i,j_A} \left(p_{j_B,i,j_A}\right)^{\widehat{n}_{j_B,i,j_A}}, \tag{B4}$$

where the relative frequencies $\widehat{n}_{j_B,i,j_A}$'s have to satisfy the following constraints:

$$\begin{aligned} &\sum_{j_B,i,j_A} n_{j_B,i,j_A} = m \\ &l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = \sum_{j_B,i,j_A} n_{j_B,i,j_A} \ln q_{j_B,i,j_A}. \end{aligned} \tag{B5}$$

By combining together the constraints (B5), we obtain a unique constraint equation for $l_{\theta_{m,j}}$, i.e.,

$$l_{\theta_{m,j}}(\vec{o},\vec{\lambda}) = m \ln q_{d_\pi,d_\lambda,d_\pi} - \widetilde{\sum_{j_B,i,j_A}} \widehat{n}_{j_B,i,j_A} \gamma_{j_B,i,j_A}, \tag{B6}$$

where

$$\gamma_{j_B,i,j_A} \equiv \frac{\ln q_{d_\pi,d_\lambda,d_\pi}}{\ln q_{j_B,i,j_A}}. \tag{B7}$$

It is worth observing that in deriving the constraint (B6) we have chosen $(d_\pi, d_\lambda, d_\pi)$ as the *reference triplet* of the configuration space that defines the stochastic trajectory of the system dynamics in a single realization. This choice is arbitrary and represents a degree of freedom of the procedure. However, that is not surprising because the number of constraints of formula (B5) is smaller than the number of relative frequencies n_{j_B,i,j_A}, so that the values of $\widehat{n}_{j_B,i,j_A}$ that satisfy Eq. (B4) are generally not uniquely determined. This means that in order to obtain a unique analytical expression

of $\mathrm{Prob}[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$ we need to answer the following question: Which are the *unique* values of the relative frequencies $\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}$ obeying the constraint equation (B6)? By generalizing the results in [33], we can prove that *there exists a unique value for the* $\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}$*'s, under the hypothesis of a sufficiently large number m of intermediate projective measurements*. To see this, let us consider the product $\widehat{n}_{a,b,c}\gamma_{a,b,c}$ with generic indices (a,b,c):

$$\widehat{n}_{a,b,c}\gamma_{a,b,c} = m\big[\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big] - \widetilde{\sum_{j_\mathrm{B},i,j_\mathrm{A};\,(j_\mathrm{B},i,j_\mathrm{A})\neq(a,b,c)}} \widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\gamma_{j_\mathrm{B},i,j_\mathrm{A}}, \quad \text{(B8)}$$

i.e.,

$$\frac{\widehat{n}_{a,b,c}}{m} = \frac{\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m}{\gamma_{a,b,c}} - \widetilde{\sum_{j_\mathrm{B},i,j_\mathrm{A};\,j_\mathrm{B}\neq a,i\neq b,j_\mathrm{A}\neq c}} \widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\frac{\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}}{m}\frac{\gamma_{j_\mathrm{B},i,j_\mathrm{A}}}{\gamma_{a,b,c}}. \quad \text{(B9)}$$

$\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}$'s being relative frequencies, it still holds that

$$\lim_{m\to\infty}\frac{\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}}{m} = 0 \quad \text{(B10)}$$

for each triplet $(j_\mathrm{B}, i, j_\mathrm{A})$ except (a,b,c), with the result that

$$\frac{\widehat{n}_{a,b,c}}{m} \asymp \frac{\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m}{\gamma_{a,b,c}} = \frac{1}{N_\mathrm{sf}}\left(\frac{\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m}{\gamma_{a,b,c}}\right), \quad \text{(B11)}$$

with N_sf denoting the corresponding *scaling factor*. Thus, this means that for *large m*

$$\widehat{n}_{a,b,c}\gamma_{a,b,c} \asymp \frac{m\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})}{N_\mathrm{sf}} = \mathrm{const} \quad \text{(B12)}$$

for each triplet (a,b,c). Equation (B12) is a quite important result, since it denotes the existence of a *unique nonequilibrium weighted partition* of the system configuration space, once we have fixed the reference triplet $(d_\pi, d_\lambda, d_\pi)$. Such a property is thus the key point for the derivation of $\mathrm{Prob}[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$. In particular, by summing together the terms $\widehat{n}_{a,b,c}\gamma_{a,b,c}$ over all the possible values that can be assumed by (a,b,c) except for the triplet $(d_\pi, d_\lambda, d_\pi)$, we find that

$$\widetilde{\sum_{j_\mathrm{B},i,j_\mathrm{A}}} \widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\gamma_{j_\mathrm{B},i,j_\mathrm{A}} \asymp \frac{(d_\mathrm{tot}-1)}{N_\mathrm{sf}}\big[m\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big], \quad \text{(B13)}$$

where $d_\mathrm{tot} \equiv 2d_\pi + d_\lambda$ is the dimension of the statistical ensemble defining the stochastic transformation of the system from t_0 to t_m. Therefore, by comparing Eqs. (B6) and (B13), one can state that $N_\mathrm{sf} = d_\mathrm{tot} - 1$, so that for m sufficiently large

$$\widehat{n}_{a,b,c} \asymp \frac{m\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})}{(d_\mathrm{tot}-1)\gamma_{a,b,c}}. \quad \text{(B14)}$$

Once we have obtained a closed solution for the value of the relative frequencies obeying the constraints (B5), we are able to validate the exponential approximation given by the *large-deviation principle* for the probability distribution of $l_{\theta_{m,j}}$ in the thermodynamic limit of $m\to\infty$. To practically derive the LD form of $\mathrm{Prob}[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$, just take Eq. (B4) and apply the Stirling approximation on $\ln(m!)$ and $\ln(\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}!)$, which is valid again in the limit of large m:

$$\begin{aligned}\mathrm{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] &= \exp\left(\ln(m!) - \sum_{j_\mathrm{B},i,j_\mathrm{A}}\ln\big(\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}!\big) + \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\ln p_{j_\mathrm{B},i,j_\mathrm{A}}\right)\\ &\asymp \exp\left(m\ln m - m - \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\ln\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}} + \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}} + \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\ln p_{j_\mathrm{B},i,j_\mathrm{A}}\right)\\ &= \exp\left(m\ln m - \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\ln\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}} + \sum_{j_\mathrm{B},i,j_\mathrm{A}}\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}\ln p_{j_\mathrm{B},i,j_\mathrm{A}}\right).\end{aligned} \quad \text{(B15)}$$

Then, by substituting the expression of $\widehat{n}_{j_\mathrm{B},i,j_\mathrm{A}}$'s given by Eq. (B14), after straightforward calculations one finds that

$$\mathrm{Prob}\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})\big] \asymp \exp\big\{-m\,I\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big]\big\}, \quad \text{(B16)}$$

where

$$I\big[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m\big] = \sum_{j_\mathrm{B}=1}^{d_\pi}\sum_{i=1}^{d_\lambda}\sum_{j_\mathrm{A}=1}^{d_\pi} f_{j_\mathrm{B},i,j_\mathrm{A}}\ln\left(\frac{f_{j_\mathrm{B},i,j_\mathrm{A}}}{p_{j_\mathrm{B},i,j_\mathrm{A}}}\right) \quad \text{(B17)}$$

is the rate function associated to the probability distribution $\mathrm{Prob}[l_{\theta_{m,j}}(\vec{o},\vec{\lambda})]$. In particular, in Eq. (B17),

$$f_{j_\mathrm{B},i,j_\mathrm{A}} \equiv \frac{\ln q_{d_\pi,d_\lambda,d_\pi} - l_{\theta_{m,j}}(\vec{o},\vec{\lambda})/m}{(d_\mathrm{tot}-1)\gamma_{j_\mathrm{B},i,j_\mathrm{A}}} \quad \text{(B18)}$$

for each triplet $(j_\mathrm{B}, i, j_\mathrm{A}) \neq (d_\pi, d_\lambda, d_\pi)$, while

$$f_{d_\pi,d_\lambda,d_\pi} \equiv 1 - \widetilde{\sum_{j_\mathrm{B},i,j_\mathrm{A}}} f_{j_\mathrm{B},i,j_\mathrm{A}}. \quad \text{(B19)}$$

As a final remark, given the generic triplet (a, b, c), it is worth noting that $f_{a,b,c}$ and the relative frequency $\widehat{n}_{a,b,c}$ are simply related by the following equation:

$$f_{a,b,c} = \frac{\widehat{n}_{a,b,c}}{m}. \tag{B20}$$

APPENDIX C: MOST PROBABLE DISTRIBUTION $l^{\star}_{\theta_{m,j}}$

Here, the most probable value $l^{\star}_{\theta_{m,j}}$ of the log-conditional probability $l_{\theta_{m,j}}(\vec{o}, \vec{\lambda})$ is derived. The latter is obtained by evaluating the value at which the rate function $I(l_{\theta_{m,j}}/m)$ of Eq. (B17) is minimized as a function of $l_{\theta_{m,j}}$ in the thermodynamic limit of $m \to \infty$. $I(l_{\theta_{m,j}}/m)$ is a positive and convex function (see Appendix B) and $l_{\theta_{m,j}}$ is given by a convex sum of the dynamical transition probabilities $q_{j_{\rm B},i,j_{\rm A}}$. Thus, a sufficient condition for its minimization is that the identities

$$\left.\frac{\partial I[l_{\theta_{m,j}}(\vec{o}, \vec{\lambda})/m]}{\partial \ln q_{j_{\rm B},i,j_{\rm A}}}\right|_{l_{\theta_{m,j}}(\vec{o},\vec{\lambda})=l^{\star}_{\theta_{m,j}}} = 0 \tag{C1}$$

[computed in correspondence to $l_{\theta_{m,j}}(\vec{o}, \vec{\lambda}) = l^{\star}_{\theta_{m,j}}$] are all simultaneously verified for *each* triplet $(j_{\rm B}, i, j_{\rm A})$. If we perform the derivative of $I(l_{\theta_{m,j}}/m)$ with respect to $\ln q_{j_{\rm B},i,j_{\rm A}}$, then we find that the identities (C1) for the triplets $(j_{\rm B}, i, j_{\rm A})$ apart from $(d_\pi, d_\lambda, d_\pi)$ can be equivalently written by means of a unique equation, i.e.,

$$p_{d_\pi,d_\lambda,d_\pi} f_{j_{\rm B},i,j_{\rm A}} = p_{j_{\rm B},i,j_{\rm A}}\left(1 - \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} f_{j_{\rm B},i,j_{\rm A}}\right). \tag{C2}$$

By summing both sides over $(j_{\rm B}, i, j_{\rm A})$, we get

$$p_{d_\pi,d_\lambda,d_\pi} \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} f_{j_{\rm B},i,j_{\rm A}} = \left(1 - \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} f_{j_{\rm B},i,j_{\rm A}}\right) \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} p_{j_{\rm B},i,j_{\rm A}}, \tag{C3}$$

which, by using $\sum_{j_{\rm B},i,j_{\rm A}} p_{j_{\rm B},i,j_{\rm A}} = 1$, gives

$$\widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} f_{j_{\rm B},i,j_{\rm A}} = \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} p_{j_{\rm B},i,j_{\rm A}}. \tag{C4}$$

It is worth observing that Eq. (C4) represents the condition for the minimization of the rate function $I[l_{\theta_{m,j}}(\vec{o}, \vec{\lambda})/m]$ with respect to $l_{\theta_{m,j}}$. This means that the most probable trajectories of the system dynamics are those allowing for the equality between the summations of the relative frequencies $f_{j_{\rm B},i,j_{\rm A}}$ and the occurrence probabilities $p_{j_{\rm B},i,j_{\rm A}}$, respectively. Therefore, this also implies that in general the same value of $l^{\star}_{\theta_{m,j}}$ can be obtained by more than one trajectory within the configuration space of the system, each of them corresponding to a different realization of the stochastic dynamics of the system.

By combining Eq. (C4) with the expression of $l_{\theta_{m,j}}(\vec{o}, \vec{\lambda}) = \sum_{j_{\rm B},j_{\rm A}=1}^{d_\pi} \int_\lambda n_{j_{\rm B},j_{\rm A}}(\lambda) \ln q_{j_{\rm B},j_{\rm A}}(\lambda) d\lambda$, using Eqs. (B7) and (B18) and substituting $l_{\theta_{m,j}}(\vec{o}, \vec{\lambda})$ with $l^{\star}_{\theta_{m,j}}$, one has that

$$\left(\ln q_{d_\pi,d_\lambda,d_\pi} - \frac{l^{\star}_{\theta_{j,m}}}{m}\right) = (d_{\rm tot} - 1)\left(1 - \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} p_{j_{\rm B},i,j_{\rm A}}\right) \frac{p_{j_{\rm B},i,j_{\rm A}}}{p_{d_\pi,d_\lambda,d_\pi}} \gamma_{j_{\rm B},i,j_{\rm A}} \tag{C5}$$

for each triplet $(j_{\rm B}, i, j_{\rm A}) \neq (d_\pi, d_\lambda, d_\pi)$. If we extend Eq. (C4) by assuming that $f_{j_{\rm B},i,j_{\rm A}} = p_{j_{\rm B},i,j_{\rm A}}\ \forall (j_{\rm B}, i, j_{\rm A})$, then

$$\frac{p_{j_{\rm B},i,j_{\rm A}} \gamma_{j_{\rm B},i,j_{\rm A}}}{p_{d_\pi,d_\lambda,d_\pi}} = \frac{f_{j_{\rm B},i,j_{\rm A}} \gamma_{j_{\rm B},i,j_{\rm A}}}{p_{d_\pi,d_\lambda,d_\pi}} = \text{const}. \tag{C6}$$

Since Eqs. (C5) have to be verified for each triplet of the system configuration space, this means that Eqs. (C5) are identically equivalent to the relation

$$\frac{l^{\star}_{\theta_{m,j}}}{m} = \ln q_{d_\pi,d_\lambda,d_\pi} - \left(1 - \widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} p_{j_{\rm B},i,j_{\rm A}}\right)\left(\widetilde{\sum_{j_{\rm B},i,j_{\rm A}}} \frac{p_{j_{\rm B},i,j_{\rm A}}}{p_{d_\pi,d_\lambda,d_\pi}} \gamma_{j_{\rm B},i,j_{\rm A}}\right), \tag{C7}$$

finally providing us the analytical expression of the most probable distribution $l^{\star}_{\theta_{m,j}}$ as given by Eq. (14).

[1] K. Jacobs, *Quantum Measurement Theory and its Applications* (Cambridge University, Cambridge, England, 2014).

[2] E. B. Davies and J. T. Lewis, An operational approach to quantum probability, Commun. Math. Phys. **17**, 239 (1970).

[3] H. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, New York, 2003).

[4] A. Rivas and S. Huelga, *Open Quantum Systems: An Introduction* (Springer-Verlag, New York, 2012).

[5] S. Gherardini, C. Lovecchio, M. M. Müller, P. Lombardi, F. Caruso, and F. S. Cataliotti, Ergodicity in randomly perturbed quantum systems, Quantum Sci. Technol. **2**, 015007 (2017).

[6] M. M. Müller, S. Gherardini, and F. Caruso, Quantum Zeno dynamics through stochastic protocols, Ann. Phys. **529**, 1600206 (2017).

[7] M. A. C. Rossi, C. Foti, A. Cuccoli, J. Trapani, P. Verrucchi, and M. G. A. Paris, Effective description of the short-time dynamics in open quantum systems, Phys. Rev. A **96**, 032116 (2017).

[8] M. Campisi, P. Talkner, and P. Hänggi, Fluctuation Theorems for Continuously Monitored Quantum Fluxes, Phys. Rev. Lett. **105**, 140601 (2010).

[9] F. Thiel, E. Barkai, and D. A. Kessler, First Detected Arrival of a Quantum Walker on an Infinite Line, Phys. Rev. Lett. **120**, 040502 (2018).

[10] A. Mishra, T. Albash, and D. A. Lidar, Finite temperature quantum annealing solving exponentially small gap problem with nonmonotonic success probability, Nat. Commun. **9**, 2917 (2018).
[11] F. W. J. Hekking and J. P. Pekola, Quantum Jump Approach for Work and Dissipation in a Two-Level System, Phys. Rev. Lett. **111**, 093602 (2013).
[12] P. Facchi and S. Pascazio, Quantum Zeno Subspaces, Phys. Rev. Lett. **89**, 080401 (2002).
[13] G. S. Uhrig, Keeping a Quantum Bit Alive by Optimized pi-Pulse Sequences, Phys. Rev. Lett. **98**, 100504 (2007).
[14] G. de Lange, Z. H. Wang, D. Riste, V. V. Dobrovitski, and R. Hanson, Universal dynamical decoupling of a single solid-state spin from a spin bath, Science **330**, 60 (2010).
[15] B. Pokharel, N. Anand, B. Fortman, and D. A. Lidar, Demonstration of Fidelity Improvement Using Dynamical Decoupling with Superconducting Qubits, Phys. Rev. Lett. **121**, 220502 (2018).
[16] F. Schäfer, I. Herrera, S. Cherukattil, C. Lovecchio, F. S. Cataliotti, F. Caruso, and A. Smerzi, Experimental realization of quantum Zeno dynamics, Nat. Commun. **5**, 4194 (2014).
[17] A. Signoles, A. Facon, D. Grosso, I. Dotsenko, S. Haroche, J.-M. Raimond, M. Brune, and S. Gleyzes, Confined quantum Zeno dynamics of a watched atomic arrow, Nat. Phys. **10**, 715 (2014).
[18] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, Rev. Mod. Phys **90**, 035005 (2018).
[19] R. Kohlhaas, A. Bertoldi, E. Cantin, A. Aspect, A. Landragin, and P. Bouyer, Phase Locking a Clock Oscillator to a Coherent Atomic Ensemble, Phys. Rev. X **5**, 021011 (2015).
[20] M. Schioppo, R. C. Brown, W. F. McGrew, N. Hinkley, R. J. Fasano, K. Beloy, T. H. Yoon, G. Milani, D. Nicolodi, J. A. Sherman, N. B. Phillips, C. W. Oates, and A. D. Ludlow, Ultrastable optical clock with two cold-atom ensembles, Nat. Phot. **11**, 48 (2017).
[21] A. Chantasri, J. Dressel, and A. N. Jordan, Action principle for continuous quantum measurement, Phys. Rev. A **88**, 042110 (2013).
[22] S. J. Weber, A. Chantasri, J. Dressel, A. N. Jordan, K. W. Murch, and I. Siddiqi, Mapping the optimal route between two quantum states, Nature (London) **511**, 570 (2014).
[23] A. Chantasri and A. N. Jordan, Stochastic path-integral formalism for continuous quantum measurement, Phys. Rev. A **92**, 032125 (2015).
[24] R. Ellis, *Entropy, Large Deviations, and Statistical Mechanics* (Springer, New York, 2006).
[25] H. Touchette, The large deviation approach to statistical mechanics, Phys. Rep. **478**, 1 (2009).
[26] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications* (Springer, New York, 2010).
[27] G. Gallavotti, J. L. Lebowitz, and V. Mastropietro, Large deviation in rarefied quantum gases, J. Stat. Phys. **108**, 831 (2002).
[28] K. Netočný and F. Redig, Large deviation for quantum spin systems, J. Stat. Phys. **117**, 521 (2004).
[29] J. P. Garrahan and I. Lesanovsky, Thermodynamics of Quantum Jump Trajectories, Phys. Rev. Lett. **104**, 160601 (2010).
[30] I. Lesanovsky, M. van Horssen, M. Guţă, and J. P. Garrahan, Characterization of Dynamical Phase Transitions in Quantum Jump Trajectories beyond the Properties of the Stationary State, Phys. Rev. Lett. **110**, 150401 (2013).
[31] S. Pigeon, L. Fusco, A. Xuereb, G. De Chiara, and M. Paternostro, Thermodynamics of trajectories of a quantum harmonic oscillator coupled to N baths, Phys. Rev. A **92**, 013844 (2015).
[32] M. Collura and F. H. L. Essler, How order melts after quantum quenches, arXiv:1901.04402.
[33] S. Gherardini, S. Gupta, F. S. Cataliotti, A. Smerzi, F. Caruso, and S. Ruffo, Stochastic quantum zeno by large deviation theory, New J. Phys. **18**, 013048 (2016).
[34] S. Gherardini, Noise as a resource. Ph.D. thesis, University of Florence, 2018, arXiv:1805.01800.
[35] The introduction of semiclassical fluctuations during the evolution of the system is responsible for the modeling of the effects on its reduced dynamics given by tracing out the environmental degrees of freedom [7]. Instead, the local couplings between the system and the environment are described by a sequence of projective measurements [34].
[36] J. J. Sakurai, *Modern Quantum Mechanics* (Addison Wesley Longman, Revised Edition, 1994).
[37] F. Caruso, V. Giovannetti, C. Lupo, and S. Mancini, Quantum channels and memory effects, Rev. Mod. Phys. **86**, 1203 (2014).
[38] S. Gherardini, L. Buffoni, M. M. Mueller, F. Caruso, M. Campisi, A. Trombettoni, and S. Ruffo, Nonequilibrium quantum-heat statistics under stochastic projective measurements, Phys. Rev. E **98**, 032108 (2018).
[39] B. Mukherjee, K. Sengupta, and S. N. Majumdar, Quantum dynamics with stochastic reset, Phys. Rev. B **98**, 104309 (2018).
[40] The contraction principle allows us to put in relation the probability distributions in LD form of two distinct quantities, one as a function of the other, by means of a continuous function.
[41] R. Wong, *Asymptotic Approximations of Integrals: Computer Science and Scientific Computing* (Academic, New York, 1989).
[42] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).