Large deviation theory:

- Sanov theorem - Jaynes Max Ent principle

n i.i.d ω_i random variables with values in a set Λ with prob. ρ_j $1 \leq j \leq |\Lambda|$

Empirical vector $L_{n,j}(\omega) = \frac{1}{n} \sum_{1}^{n}{}_i \, \delta_{\omega_i, j}$

Gärtner-Ellis theorem for random vectors, $k \in \mathbb{R}^{|\Lambda|}$

scaled cumulant generating function (k is a vector with $|\Lambda|$ components)

$$\lambda(k) = \lim_{n \to \infty} \frac{1}{n} \ln \langle \exp(n k \cdot L_n) \rangle$$

$$= \ln \sum_{1}^{|\Lambda|}{}_j \, \rho_j \exp(k_j)$$

$\lambda(k)$ analytic $\forall k \in \mathbb{R}^{|\Lambda|}$ $\Rightarrow$

we can apply the G-E th $\Rightarrow$

L.D. principle holds for $L_n(\omega)$ with a rate function $I(\ell)$

$$I(\ell) = \sup_k [k \cdot \ell - \lambda(k)]$$

$\nabla_i k_j = \delta_{ij}$

$$\to \quad = k^*(\ell) \cdot \ell - \lambda(k^*) \quad \text{with } k^* / \underbrace{\nabla \lambda = \ell}$$

$$\nabla \lambda(k) = \frac{\sum_j \rho_j \exp(k_j) \nabla k_j}{\sum_\ell \rho_\ell \exp(k_\ell)} = \frac{\rho \exp(k)}{\sum_\ell \rho_\ell \exp(k_\ell)}$$

$$\underline{\nabla \lambda = \ell} \longrightarrow \frac{\rho_j \exp(k_j^*)}{\underbrace{\sum_\ell \rho_\ell \exp(k_\ell^*)}_{\exp[\lambda(k^*)]}} = \ell_j$$

$$\Downarrow$$

$$k_j^* = \ln\left(\ell_j/\rho_j\right) + \lambda(k^*)$$

$$I(\ell) = k^*\cdot\ell - \lambda(k^*) = \sum_j \ell_j \ln\left(\ell_j/\rho_j\right) + \underbrace{\Big(\sum_j \ell_j\Big)}_{1} \cancel{\lambda(k^*)} - \cancel{\lambda(k^*)}$$

$$\boxed{I(\ell) = \sum_j \ell_j \ln\left(\ell_j/\rho_j\right) = D[\ell/\rho]} \longrightarrow \text{Sanov theorem}$$

Easily generalised to $\mathbb{R}$ continuous "empirical vector"

$$L_n(x) = \frac{1}{n}\sum_{i=1}^{n} \delta(w_i - x) \qquad x \in \mathbb{R}$$

w distributed according to $\rho(x)$

$$I(\mu) = D[\mu/\rho] = \int_{-\infty}^{+\infty} dx\, \mu(x) \ln\left[\frac{\mu(x)}{\rho(x)}\right]$$

Jaynes Max Ent. principle

n particles with single particle energies $\varepsilon_1, \dots, \varepsilon_n$

Total energy $h_n = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i$ (energy density)

(non interacting particles)

One-particle energy distribution (empirical vector)

$$L_n(\varepsilon) = \frac{1}{n} \sum_{1}^{n} \delta(\varepsilon_i - \varepsilon)$$ fraction of particles with given energy

$$\tilde{h}(L_n) = \int_{\Lambda} \varepsilon L_n(\varepsilon) \, d\varepsilon$$ "contraction"

$L_n(\varepsilon) \longrightarrow$ "macrostate"

max ent principle: $$s(u) = \sup_{\mu / \tilde{h}(\mu) = u} [- I^u(\mu)]$$

Sanov theorem $\rightarrow$ $\underbrace{I^u(\mu) = -\tilde{S}(\mu) + \ln|\Lambda|}$

$$\tilde{S}(\mu) = -\int_{\Lambda} d\varepsilon \, \mu(\varepsilon) \ln[\mu(\varepsilon)]$$ Shannon entropy of $\mu(\varepsilon)$

uniform prior distribution for $\rho(\varepsilon)$

$$\boxed{s(u) = \sup_{\mu : \tilde{h}(\mu) = u} [\tilde{S}(\mu)] - \ln|\Lambda|}$$ JAYNES MAX ENT Principle