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ESERCIZIO 1

$\mathbb{R}^4$

$$U = \left\langle u = \begin{pmatrix} 12 \\ 3 \\ -2 \\ 0 \end{pmatrix} \right\rangle \qquad W: \; x_1 - 2x_2 + 3x_3 - 4x_4 = 0$$

$$x_1 = 2x_2 - 3x_3 + 4x_4$$

a) $U \subset W$, completate u ad una base di W.

$$12 - 2 \cdot 3 + 3 \cdot (-2) - 4 \cdot 0 =$$
$$12 - 6 - 6 - 0 = 0 \quad \Rightarrow \quad u \in W$$

$U \subset W$ ✓

$$W = \left\{ \begin{pmatrix} 2x_2 - 3x_3 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \right\}$$
$$= \left\langle w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

BASE

$$\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0$$

$$\begin{cases} 2\lambda_1 - 3\lambda_2 + 4\lambda_3 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases} \quad ; \quad 0 = 0$$

Verifichiamo se u, w_1, w_3 e' base

$$\lambda_1 u + \lambda_2 w_1 + \lambda_3 w_3 = 0$$

$$\begin{cases} 12\lambda_1 + 2\lambda_2 + 4\lambda_3 = 0 \\ 3\lambda_1 + \lambda_2 = 0 \\ -2\lambda_1 = 0 \\ \lambda_3 = 0 \end{cases} \quad \begin{cases} 0 = 0 \\ \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_3 = 0 \end{cases}$$

$$B = \{u, w_1, w_3\}$$

b) $V \subset \mathbb{R}^4$ $\quad V = \left\langle \underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 3 \\ 4 \\ -1 \end{pmatrix}} \right\rangle$

$V \cap W$

$V + W$

w un generico vettore di W

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 2\lambda_1 - 3\lambda_2 + 4\lambda_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

v un generico vettore di V

$$a \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 3 \\ 4 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} a + 2b \\ 2a + 3b \\ 3a + 4b \\ -b \end{pmatrix}$$

$$a\begin{pmatrix}2\\3\\0\end{pmatrix}+b\begin{pmatrix}3\\4\\-1\end{pmatrix} \longrightarrow \begin{pmatrix}3a+4b\\-b\end{pmatrix}$$

$$\begin{cases} a+2b = 2\lambda_1 - 3\lambda_2 + 4\lambda_3 \\ 2a+3b = \lambda_1 \\ 3a+4b = \lambda_2 \\ -b = \lambda_3 \end{cases} \qquad \begin{cases} a+2b = 4a+6b-9a-12b-4b \\ \lambda_1 = 2a+3b \\ \lambda_2 = 3a+4b \\ \lambda_3 = -b \end{cases}$$

$$\begin{cases} a-4a+9a = -2b+6b-12b-4b \\ * \\ * \\ * \end{cases} \qquad \begin{cases} 6a = -12b \\ * \\ * \\ * \end{cases}$$

$$\begin{cases} a = -2b \\ \lambda_1 = 2(-2b)+3b \\ \lambda_2 = 3(-2b)+4b \\ \lambda_3 = -b \end{cases} \qquad \begin{cases} a = -2b \\ \lambda_1 = -b \\ \lambda_2 = -2b \\ \lambda_3 = -b \end{cases}$$

$$v \in V \cap W \qquad v = \begin{pmatrix} -2b+6b-4b \\ -b \\ -2b \\ -b \end{pmatrix} = \begin{pmatrix} 0 \\ -b \\ -2b \\ -b \end{pmatrix}$$

$$V \cap W = \left\langle \begin{pmatrix} 0 \\ -1 \\ -2 \\ -1 \end{pmatrix} \right\rangle$$

; COEF.

$$\dim(V+W) = \dim V + \dim W - \dim(V \cap W)$$
$$= 2 + 3 - 1 = 4$$

$$V + W = \mathbb{R}^4$$

c) $v_t = \begin{pmatrix} t \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad t \in \mathbb{R}$

v_1, v_2, v_t SONO DIPENDENTI $\quad [t = -1]$

$$v_t = \lambda_1 v_1 + \lambda_2 v_2$$

PRODOTTO TRA MATRICI

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & -1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 2 & 5 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

APPLICAZIONI LINEARI

$$f : \mathbb{R}^3 \to \mathbb{R}^4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_3 \\ x_1 + x_2 \\ x_2 + x_3 \\ \end{pmatrix} \nearrow \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \end{pmatrix} \quad 1 \; 2 \; 1$$

a) matrice associata

$$f(e_1) = f\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$f(e_2) = f\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$f(e_3) = f\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$M^{\mathcal{E}}_{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \qquad \begin{aligned} \mathcal{B} &= \{e_1, e_2, e_3\} \\ \mathcal{E} &= \{e_1, e_2, e_3, e_4\} \end{aligned}$$

b) $Im(f)$, $Ker(f)$

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

BASE?

$$Ker(f) = \left\{ \begin{pmatrix} x_1 \\ \end{pmatrix} \in \mathbb{R}^3 \mid f\left[\begin{pmatrix} x_1 \\ \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\mathrm{Ker}(f) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; f\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = x_3 \\ \cancel{x_3} - \cancel{x_3} = 0 \\ x_2 = -x_3 \\ \cancel{x_3} - 2\cancel{x_3} + \cancel{x_3} = 0 \end{cases}$$

$$\mathrm{Ker}(f) = \left\{ \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

ESERCIZIO

$f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad \mathcal{B} = \{e_1, e_2, e_3\} \quad \mathcal{C} = \{e_1, e_2\}$

$$f(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f(e_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

a) $$M_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_3 \\ x_2 + x_3 \end{pmatrix}$$

$$\mathrm{Im}(f) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{Ker}(f) : \quad \begin{cases} x_1 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \end{cases}$$

$$\mathrm{Ker}(f) = \left\langle \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\rangle \qquad \left\{ \begin{pmatrix} -2x_3 \\ -x_3 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\}$$