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ES 1

$\mathbb{R}^4 \quad \mathcal{E}_4 = \{e_1, e_2, e_3, e_4\}$

U sottospazio generato da u_1, u_2, u_3

$u_1 = 2e_1 - e_2 + e_3 \qquad u_2 = e_2 - e_3 + 2e_4$

$u_3 = e_1 + e_4$

W sottospazio generato da w_1, w_2, w_3, w_4

$w_1 = e_1 + 2e_2 \qquad w_2 = e_2 - e_4 \qquad w_3 = e_2 + e_3 - e_4$

$w_4 = e_1 - e_3$

1) TROVARE UNA BASE PER U e W

$$u_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$$

$$\begin{cases} 2\lambda_1 + \lambda_3 = 0 \\ -\lambda_1 + \lambda_2 = 0 \\ \lambda_1 - \lambda_2 = 0 \\ 2\lambda_2 + \lambda_3 = 0 \end{cases} \qquad \begin{cases} \lambda_3 = -2\lambda_1 \\ \lambda_2 = \lambda_1 \\ 0 = 0 \\ \cancel{2\lambda_1} - \cancel{2\lambda_1} = 0 \end{cases}$$

SONO DIP.

$0 = 0$

ESCLUDIAMO u_1, VERIFICHIAMO SE

$$\lambda_1 u_2 + \lambda_2 u_3 = 0$$

$$\begin{cases} \lambda_2 = 0 \\ \lambda_1 = 0 \\ -\lambda_1 = 0 \\ 2\lambda_1 + \lambda_2 = 0 \end{cases} ; \quad 0=0$$

SONO INDIP.

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad w_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad w_4 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

...... Base w_1, w_2, w_3

$$\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \lambda_4 w_4 = 0$$

ESCLUDO w_4

$$\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0$$

$\lambda_1 = 0$
$\lambda_2 = 0$
$\lambda_3 = 0$
INDIP.

2) $f : \mathbb{R}^4 \to \mathbb{R}^4$ appl. lineare

$f(e_1) = e_1 + e_3$
$f(e_2) = e_2 + e_3$
$f(e_3) = e_1$
$f(e_4) = e_2$

Determina dimens. e base di
$f(U)$
$f(W)$

$$M_{\mathcal{E}_4}^{\mathcal{E}_4}(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f(e_1) = e_1 + e_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad f(e_3) = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f(e_2) = e_2 + e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad f(e_4) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \\ x_1 + x_2 \\ 0 \end{pmatrix}$$

$$f(U) = \langle f(u_2), f(u_3) \rangle$$

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f\left[\begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} 0 - 1 \\ 1 + 2 \\ 0 + 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$f(u_2) = M_{\mathcal{E}_4}^{\mathcal{E}_4}(f) \cdot u_2$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ \cdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ \cdots \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ \cdots \end{pmatrix}$$

$$= \begin{pmatrix} [illegible] & [illegible] & [illegible] & [illegible] \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} [illegible] \\ 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} [illegible] \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$f(u_3) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$f(U) = \left\langle \begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$ INDIP

$\dim(f(U)) = 2$

$$f(W) = \langle f(w_1), f(w_2), f(w_3) \rangle$$

$$= \left\langle \underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}} \right\rangle$$

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$$

DIP.

$$\begin{cases} \lambda_1 + \lambda_3 = 0 \\ 2\lambda_1 + 2\lambda_3 = 0 \\ 3\lambda_1 + \lambda_2 + \lambda_3 = 0 \\ 0 = 0 \end{cases} \qquad \begin{cases} \lambda_1 = -\lambda_3 \\ 0 = 0 \\ -3\lambda_3 + \lambda_2 + \lambda_3 = 0 \\ 0 = 0 \end{cases} \qquad \begin{cases} \lambda_1 = -\lambda_3 \\ 0 = 0 \\ \lambda_2 = 2\lambda_3 \\ 0 = 0 \end{cases}$$

$$f(W) = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$ INDIP

$\dim(f(W)) = 2$

c) Dove $M^{\mathcal{E}_4}_{\mathcal{E}_4}(f)$

d) $B = \{v_1, v_2, v_3, v_4\}$ base di $\mathbb{R}^4$

$v_1 = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_2 = e_1 + e_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = e_3 + e_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_4 = e_2 + e_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$M^{B}_{\varepsilon_4}(f) = ?$

COLONNE $f(e_1), f(e_2), f(e_3), f(e_4)$
ESPRESSI COME COORDINATE DI B.

$$f(e_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 = \lambda_2 \\ 0 = \lambda_4 \\ 1 = \lambda_2 + \lambda_3 + \lambda_4 \\ 0 = \lambda_1 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_2 = 1 \\ \lambda_4 = 0 \\ \cancel{1} = \cancel{1} + \lambda_3 + 0 \\ \lambda_1 = 0 \end{cases} \qquad \lambda_3 = 0$$

$(0; 1; 0; 0)$ PRIMA COLONNA

$$f(e_2) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$(0; 0; 0; 1)$ SECONDA COLONNA

$$f(e_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} 1 = \lambda_2 \\ 0 = \lambda_4 \\ 0 = \lambda_2 + \lambda_3 + \lambda_4 \\ 0 = \lambda_1 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_2 = 1 \\ \lambda_4 = 0 \\ \lambda_3 = -1 \\ \lambda_1 = 1 \end{cases}$$

$(1, 1, -1, 0)$ TERZA COLONNA

$$f(e_4) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = \lambda_2 \\ 1 = \lambda_4 \\ 0 = \lambda_2 + \lambda_3 + \lambda_4 \\ 0 = \lambda_1 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_2 = 0 \\ \lambda_4 = 1 \\ \lambda_3 = -1 \\ \lambda_1 = 1 \end{cases}$$

$(1; 0; -1; 1)$ QUARTA COLONNA

$$M^{B}_{\mathcal{E}_4}(f) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

OPPURE:

$\rightarrow \; M^{\mathcal{E}'}_{B'}(f) = M^{\mathcal{E}'}_{\mathcal{E}}(id) \cdot M^{\mathcal{E}}_{B}(f) \cdot M^{B}_{B'}(id)$

$\rightarrow \; A' = S \cdot A \cdot P^{-1}$

$\rightarrow \quad A' = S \cdot A \cdot P^{-1}$

$P = M_{B}^{B'}(id)$

$M_{\mathcal{E}_4}^{B}(f) = \underline{M_{\mathcal{E}_4}^{B}(id)} \cdot M_{\mathcal{E}_4}^{\mathcal{E}_4}(f) \cdot \cancel{M_{\mathcal{E}_4}^{\mathcal{E}_4}(id)}$

$M_{\mathcal{E}_4}^{\mathcal{E}_4}(id)$:

$id(e_1) = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad id(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$id(e_3) = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad id(e_4) = e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$M_{\mathcal{E}_4}^{\mathcal{E}_4}(id) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ← E' ELEM. NEUTRO DELLA MOLTIPL.

$\mathbb{R}^2 \qquad B = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$

$M_{B}^{B}(id) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$id(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 \qquad (1, 0)$

$id(v_2) = v_2 = 0 \cdot v_1 + 1 \cdot v_2 \qquad (0; 1)$

$M_{\mathcal{E}_4}^{B}(id)$:

$$id(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightarrow (1, 1, -1, 0)$$

$$id(e_2) = e_2 \longrightarrow (1; 0; -1, 1)$$

$$id(e_3) = e_3 \longrightarrow (-1, 0; 1, 0)$$

$$id(e_4) = e_4 \longrightarrow (1; 0; 0; 0)$$

$$M^{B}_{\mathcal{E}_4}(f) = \underbrace{\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{M^{B}_{\mathcal{E}_4}(id)} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{M^{\mathcal{E}_4}_{\mathcal{E}_4}(f)} = \ldots \ldots$$

- $M^{\mathcal{E}_4}_{B}(f)$
- $M^{B}_{B}(f)$

PER ALLENAMENTO, CON ENTRAMBI i METODI