2021-11-26

venerdì 26 novembre 2021 16:32

RIDUZIONI E RANGO DI MATRICI

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & 2 \\ -1 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[III+I]{II-2I} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{III+2II} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 4 & 5 \end{pmatrix} \qquad rK = 3$$

SCALA RIDOTTA:

$$\xrightarrow[\frac{1}{4}(III)]{(-1)II} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 5/4 \end{pmatrix}$$

$$-2 + \cancel{2}\frac{5}{\cancel{4}_2} = \frac{1}{2}$$

$$\xrightarrow[I+III]{II+2III} \begin{pmatrix} 1 & 2 & 0 & 5/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 5/4 \end{pmatrix}$$

$$\frac{5}{4} - \cancel{2}\frac{1}{\cancel{2}} = \frac{1}{4}$$

$$\xrightarrow{I-2II} \begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 5/4 \end{pmatrix}$$

ES 2 RANGO AL VARIARE DI $a \in \mathbb{R}$

$$A = \begin{pmatrix} 1 & a & 0 & 0 & 1 \\ -1 & 1-a & 2 & 1 & -1 \\ 0 & 1 & a+2 & a & 2-a \\ 1 & a & 0 & a-1 & 2-a \end{pmatrix}$$

$$\xrightarrow[\text{III}-\text{I}]{\text{II}+\text{I}} \begin{pmatrix} 1 & a & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & a+2 & a & 2-a \\ 0 & 0 & 0 & a-1 & 1-a \end{pmatrix}$$

$$\xrightarrow{\text{III}-\text{II}} \begin{pmatrix} 1 & a & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & a & a-1 & 2-a \\ 0 & 0 & 0 & a-1 & 1-a \end{pmatrix}$$

$rk = 4$ se i PIVOT $\neq 0$

$\hookrightarrow a \neq 0 \quad a-1 \neq 0 \rightarrow a \neq 1$

se $a = 0$:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{IV}-\text{III}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad rk = 4$$

Se $a=1$: $A=\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$rk = 3$

- $A = \begin{pmatrix} 1 & 0 & 1 & k \\ k & k & 2+k & k^2+k \\ -1 & k & k & 0 \\ 0 & -k^2 & -k-1 & -1 \end{pmatrix}$ $rk = 4$ se $k \neq 0, \pm 1$

se $k=1 \Rightarrow rk = 2$
se $k=-1 \Rightarrow rk = 3$
se $k=0 \Rightarrow rk = 3$

DETERMINANTE E INVERSA

$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$\det(A) = \det\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

n° rigo + n° colone

$= \cancel{0 \cdot (-1)^3 \det\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}} + 1 \cdot (-1)^4 \det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$

$+ \cancel{0 \cdot (-1)^5 \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$

$$= 1 \cdot 1 \cdot \det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = 1(-1) - (1)(2) = -3$$

invertibile

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{III-I} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{III-II} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{(-\frac{1}{3})III} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right) \xrightarrow{I-2III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -2/3 & 2/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & -1/3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det B = 1 \cdot (-1)^4 \det\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= 1 \cdot [2(-1) - 1(2)] = -4 \neq 0$$

invertibile

$$\xrightarrow{I \leftrightarrow III}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{I \leftrightarrow III} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[III-2I]{II+I} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 & 0 & -2 \end{array}\right) \xrightarrow{\frac{1}{4}III} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1/4 & 0 & -1/2 \end{array}\right)$$

$$\xrightarrow{I+III} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1/4 & 0 & -1/2 \end{array}\right) \qquad B^{-1} = \begin{pmatrix} 1/4 & 0 & 1/2 \\ 0 & 1 & 1 \\ 1/4 & 0 & -1/2 \end{pmatrix}$$

ES. TUTORATO PRECEDENTE

$$M_{\mathcal{E}_4}^{\mathcal{E}_4}(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \left\{ \overset{v_1}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}} \right\}$$

$$M_{\mathcal{E}_4}^{B}(f) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$M^{B}_{\mathcal{E}_4}(f) = M^{B}_{\mathcal{E}_4}(id) \cdot M^{\mathcal{E}_4}_{\mathcal{E}_4}(f) \cdot \cancel{M^{\mathcal{E}_4}_{\mathcal{E}_4}(id)}$$

$$M^{B}_{\mathcal{E}_4}(id) = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M^{B}_{\mathcal{E}_4}(id) = \Big(\underbrace{M^{\mathcal{E}_4}_{B}(id)}_{\text{FACILE}}\Big)^{-1}$$

$M^{\mathcal{E}_4}_{B}(id)$ LE SUE COLONNE SONO LE COORDINATE RISPETTO $\mathcal{E}_4$ DEI VETTORI OTTENUTI APPLICANDO id AI VETTORI DI B

$$id(v_1) = v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$ PRIMA COLONNA

$$id(v_2) = v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ SECONDA COLONNA

v_1, v_2, v_3 e v_4 SONO LE COLONNE $M^{\mathcal{E}_4}_{B}(id)$

$$M^{\mathcal{E}_4}_{B}(id) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I \leftrightarrow IV} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{II \leftrightarrow IV} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{III - II} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{III - IV} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{I - III} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$M_B^{\mathcal{E}_4}(id)$

APPLICAZIONI LINEARI

$$f: \mathbb{R}[x]_{\leq 3} \longrightarrow \mathbb{R}^3$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \longmapsto \begin{pmatrix} a_0 + 2a_2 \\ a_1 - a_2 + a_3 \\ a_0 + a_1 + a_2 + a_3 \end{pmatrix}$$

Nelle basi canoniche $\mathcal{B} = \{1, x, x^2, x^3\}$

$\mathcal{E}_3 = \{e_1, e_2, e_3\}$

1) Determinare $Ker(f)$ e $Im(f)$

$$\begin{cases} a_0 + 2a_2 = 0 \\ a_1 - a_2 + a_3 = 0 \\ a_0 + a_1 + a_2 + a_3 = 0 \end{cases} \qquad \begin{cases} a_0 = -2a_2 \\ a_1 = a_2 - a_3 \\ \cancel{-2a_2} + \cancel{a_2} - \cancel{a_3} + \cancel{a_2} + \cancel{a_3} = 0 \quad 0 = 0 \end{cases}$$

$$-2a_2 + (a_2 - a_3)x + a_2 x^2 + a_3 x^3$$

$$a_2 \underbrace{(-2 + x + x^2)} + a_3 \underbrace{(-x + x^3)}$$

$$Ker(f) = \langle -2 + x + x^2, -x + x^3 \rangle$$

$$M_{\mathcal{B}}^{\mathcal{E}_3}(f) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\rangle$$

1) $\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$

2) GUARDO SE VEDO "AD OCCHIO"

2) GUARDO SE VEDO "AD OCCHIO"

3) SCRIVO i VETTORI COME **RIGHE** DI UNA MATRICE E LA RIDUCO IN SCALA

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

LA RIDUCO, LE RIGHE CHE MI RESTANO DALLA RIDUZIONE SONO UNA BASE

$$\xrightarrow{III-2I} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{III+II} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Im(f) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$