ALGEBRA LINEARE E GEOMETRIA

Lezioni 52-53, 06/12/2021

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Richiamo dalla lezione precedente:

- A matrice $n \times n$, λ autovalore

$$1 \leq m.g.(\lambda) \leq m.a.(\lambda) \leq n$$

- Teorema di diagonalizzabilità: $A \in Mat_{n\times n}(\mathbb{K})$ è diagonalizzabile $\Longleftrightarrow \begin{cases} 1) \text{ Il polinomio caratteristico di } A \text{ ammette tutte le radici in } \mathbb{K}. \\ 2) \text{ m.g.} = \text{m.a. per ogni autovalore} \end{cases}$

- $\lambda = 0$ è autovalore di $A \Longleftrightarrow A$ non è invertibile.

- $A = (a_{ij}) \in Mat_{n\times n}(\mathbb{K})$ $\boxed{tr(A) = \sum_{i=1}^{n} a_{ii}}$

  - $A, B \in Mat_{n\times n}(\mathbb{K}) \Rightarrow tr(AB) = tr(BA)$.
  - A e B sono simili $\Rightarrow tr(A) = tr(B)$.

- A diagonalizzabile $\Rightarrow A$ è simile a la matrice diagonale $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$ autovalori di A.

$\Rightarrow tr(A) = \lambda_1 + \dots + \lambda_n$ = Somma degli autovalori

$det(A) = \lambda_1 \cdots \lambda_n$ = Prodotto degli autovalori.

Questo è vero in generale $\left(\begin{matrix}\text{anche se } A \text{ non} \\ \text{è diagonalizzabile}\end{matrix}\right)$

$$\boxed{\begin{aligned} tr(A) &= \lambda_1 + \dots + \lambda_n \\ \det(A) &= \lambda_1 \cdots \lambda_n \end{aligned}}$$

<u>oss</u> Gli autovalori compaiono con molteplicità e possono essere <u>complessi</u>.

<u>Esempio</u>

1) $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

$\lambda_1 = \dfrac{5 + i\sqrt{3}}{2}$

$\lambda_2 = \dfrac{5 - i\sqrt{3}}{2} = \overline{\lambda}_1$

(vedi lezioni 46-47 29/11/2021)

$$\lambda_1 + \lambda_2 = \left(\frac{5}{2} + i\frac{\sqrt{3}}{2}\right) + \left(\frac{5}{2} - \frac{i\sqrt{3}}{2}\right) = \frac{5}{2} + \frac{5}{2} = \underset{\underset{2+3}{\parallel}}{\underset{tr(A)}{\underset{\parallel}{5}}}$$

$$\lambda_1\lambda_2 = \lambda_1\overline{\lambda}_1 = |\lambda_1|^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{25}{4} + \frac{3}{4} = \frac{28}{4} = 7 = \underbrace{(2)(3) - (-1)(1)}_{\det(A)}$$

2) $A = \begin{pmatrix} 2 & 0 & 3 \\ -2 & -4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$\lambda_1 = 2$
$\lambda_2 = 2$
$\lambda_3 = -4$

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 - 4 = 0 = tr(A)$$

$$\lambda_1\lambda_2\lambda_3 = 2(2)(-4) = -16 = \det(A)$$

Alcune proprietà del polinomio caratteristico.

$$A \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \qquad \mathbb{K} = \mathbb{C} \text{ o } \mathbb{R}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= (-1)^n \lambda^n + \underbrace{(-1)^{n+1} \mathrm{tr}(A)}\lambda^{n-1} + \ldots + \ast\,\lambda + \underbrace{\det(A)}$$

$$p(0) = \det(A - 0 \cdot I) = \det(A)$$

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

$$= (-1)^n (\lambda^n + \underbrace{(-\lambda_1 - \lambda_2 - \ldots - \lambda_n)}_{-\mathrm{tr}(A)}\lambda^{n-1} + \ldots)$$

$$= (-1)^n \lambda^n + (-1)^{n+1} \mathrm{tr}(A) + \ldots$$

Esempi

1) $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ $\qquad \det(A - \lambda I) = \lambda^2 \underbrace{-5}_{-\mathrm{tr}(A)}\lambda + \underbrace{7}_{\det A}$

A matrice $n\times n$ diagonalizzabile.

$$\Rightarrow \quad A = P D P^{-1}$$

D: matrice diagonale

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$$A^2 = \overbrace{(PDP^{-1})}^{A}\overbrace{(PDP^{-1})}^{A} = PD\overbrace{(P^{-1}P)}^{I}DP^{-1}$$
$$= PDDP^{-1} = PD^2P^{-1}$$

$$A^3 = \overbrace{(PD^2P^{-1})}^{A^2}\cdot\overbrace{(PDP^{-1})}^{A} = PD^2\overbrace{(P^{-1}P)}^{I}DP^{-1}$$
$$= PD^3P^{-1}$$

$\vdots$

$$A^r = PD^rP^{-1} \qquad D^r = \begin{pmatrix} \lambda_1^r & 0 & \cdots & 0 \\ 0 & \lambda_2^r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^r \end{pmatrix}$$

$$\left(e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \right)$$

PROIEZIONI E SIMMETRIE

$$V = U \oplus W$$

$v \in V$ si scrive in modo unico $v = u + w$

$u \in U$ $\quad$ $w \in W$

Chiamiamo:

- Proiezione su U nella direzione W la funzione lineare:

$$\Pi_U^W : \begin{cases} V \longrightarrow V \\ v = u + w \longmapsto u \end{cases}$$

$$\Pi_U^W(v) = u$$

$$\pi_U^W \circ \pi_U^W = \pi_U^W$$

$$\begin{tikzcd} u+w \arrow[r, mapsto, "\pi_U^W"] \arrow[rr, bend right, "\pi_U^W"'] & u+\underset{\in W}{0} \arrow[r, mapsto, "\pi_U^W"] & u \end{tikzcd}$$

$\underline{v}$ = base di V

$$M_{\underline{v}}^{\underline{v}}(\pi_U^W)\, M_{\underline{v}}^{\underline{v}}(\pi_U^W) = M_{\underline{v}}^{\underline{v}}(\pi_U^W)$$

- Simmetria di asse U e direzione W la funzione lineare

$$\sigma_U^W : \begin{cases} V \longrightarrow V \\ v = u+w \longmapsto u-w \end{cases}$$

$$\sigma_U^W \circ \sigma_U^W = id_V$$

$$\begin{tikzcd} u+w \arrow[r, mapsto, "\sigma_U^W"] \arrow[rr, bend right, "id_V"'] & u-w \arrow[r, mapsto, "\sigma_U^W"] & u+w \end{tikzcd}$$

$$M_{\underline{v}}^{\underline{v}}(\sigma_U^W)\, M_{\underline{v}}^{\underline{v}}(\sigma_U^W) = I$$

$\underline{v}_U = \{u_1, \dots, u_r\}$ base di U

$\underline{v}_W = \{w_1, \dots, w_s\}$ base di W.

$\underline{v} = \{u_1, \dots, u_r, w_1, \dots, w_s\}$ base di V.

$$M_{\underline{v}}^{\underline{v}}(\pi_U^W) = \begin{array}{c} \begin{array}{cc} r \times r & \end{array} \\ \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & \vdots & & \vdots \\ 0 & 0 & \ddots & \vdots & & & \\ 0 & \cdots & 0 & 1 & & & \\ \hline 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & \cdots & & & 0 & \cdots & 0 \end{array}\right) \\ \begin{array}{cc} & s \times s \ (\text{zero}) \end{array} \end{array}$$

$\pi_U^W(u_j) = u_j$ — u_j è autovettore relativo a $\lambda = 1$

$\pi_U^W(w_k) = \pi_U^W(\underset{\in U}{\vec{0}} + w_k) = \vec{0}$

w_k è autovettore relativo a $\lambda = 0$
$w_k \in \ker(\pi_U^W)$.

$$M_{\underline{v}}^{\underline{v}}(\sigma_U^W) = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & 0 & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & -1 & & \vdots \\ & & & & \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -1 \end{array}\right)$$

$r \times r$ $s \times s$

$\sigma_U^W(u_j) = u_j$ u_j è autovettore relativo a $\lambda = 1$

$\sigma_U^W(w_k) = -w_k$ w_k è autovettore relativo a $\lambda = -1$.

Esempio

$V = \mathbb{R}^2 \qquad U = \langle e_1 \rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$

$W = \langle e_1 + e_2 \rangle = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$

$\underline{v} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\pi_U^W\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\pi_U^W\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$M_{\underline{v}}^{\underline{v}}(\pi_U^W) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\sigma_U^W\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\sigma_U\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$M_{\underline{v}}^{\underline{v}}(\sigma_U^W) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\underline{e} = \{e_1, e_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$M_{\underline{e}}^{\underline{e}}(\pi_U^W) = ? \qquad M_{\underline{e}}^{\underline{e}}(\sigma_U^W) = ?$

$\pi_U^W\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\pi_U^W\begin{pmatrix} 0 \\ 1 \end{pmatrix} = ? \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\pi_U^W\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = -1\, \pi_U^W\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \pi_U^W\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= -\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$M^{\underline{e}}_{\underline{e}}(\pi_U^W) = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$$

Controllo che $\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}^2 = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$:

$$\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} \checkmark$$

$$\sigma_U^W\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$\sigma_U^W\begin{pmatrix}0\\1\end{pmatrix} = \sigma_U^W\left(-1\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}1\\1\end{pmatrix}\right)$$

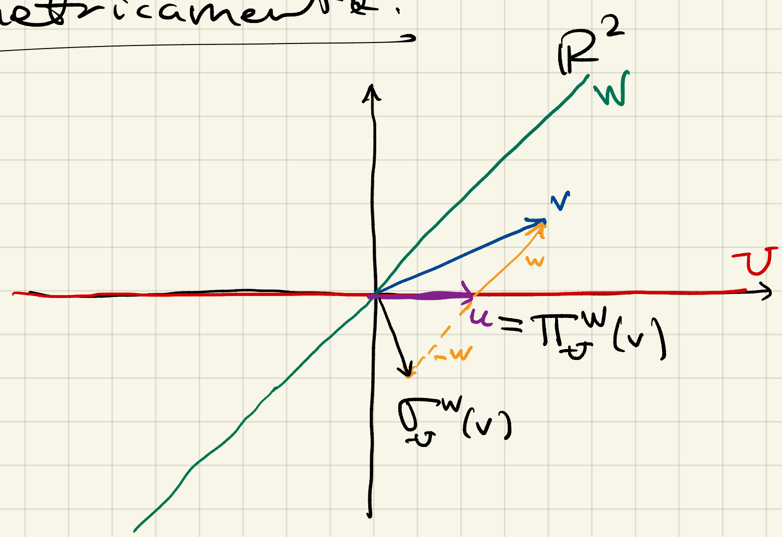
$$= -\sigma_U^W\begin{pmatrix}1\\0\end{pmatrix} + \sigma_U^W\begin{pmatrix}1\\1\end{pmatrix}$$

$$= -\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}$$

$$M^{\underline{e}}_{\underline{e}}(\sigma_U^W) = \begin{pmatrix}1 & -2\\ 0 & -1\end{pmatrix}$$

$$\begin{pmatrix}1 & -2\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \checkmark$$

Geometricamente:



$U = \left\langle \begin{pmatrix}1\\0\end{pmatrix} \right\rangle$

$W = \left\langle \begin{pmatrix}1\\1\end{pmatrix} \right\rangle$

Esempio 2 $\qquad V = \mathbb{R}^3$

$$U = \left\{ \overset{u_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\} \qquad W = \left\langle \overset{w_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \right\rangle$$

$$\underline{v} = \{u_1, u_2, w_1\} \qquad M^{\underline{v}}_{\underline{v}}(\pi^W_U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\pi^W_U(u_1) = u_1$

$\pi^W_U(u_2) = u_2$

$\pi^W_U(w_1) = \vec{0}$

$\sigma^W_U(u_1) = u_1$

$\sigma^W_U(u_2) = u_2$

$\sigma^W_U(w_1) = -w_1$

$$M^{\underline{v}}_{\underline{v}}(\sigma^W_U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

base canonica

$$\underline{e} = \left\{ \underset{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{e_1}, \underset{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{e_2}, \underset{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{e_3} \right\}$$

$$M^{\underline{e}}_{\underline{e}}(\pi^W_U) = ? \qquad M^{\underline{e}}_{\underline{e}}(\sigma^W_U) = ?$$

$$\pi^W(e_1) = e_1 = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$$

$$\pi^W_U(e_2) = \pi^W_U(w_1 - u_1) = \underbrace{\pi^W_U(w_1)}_{\vec{0}} - \underbrace{\pi^W_U(u_1)}_{u_1 = e_1}$$

$$= -e_1 = -1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$$

$$\pi^W_U(e_3) = \pi^W_U(u_2 - w_1 + u_1)$$

$$= \underbrace{\pi^W_U(u_2)}_{u_2} - \underbrace{\pi^W_U(w_1)}_{\vec{0}} + \underbrace{\pi^W_U(u_1)}_{u_1 = e_1}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e_1 + e_2 + e_3$$

$$M^{\underline{e}}_{\underline{e}}(\pi^W_U) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{\mathcal{U}}^{\mathcal{W}}(e_1) = \sigma_{\mathcal{U}}^{\mathcal{W}}(u_1) = u_1 = e_1$$

$$\sigma_{\mathcal{U}}^{\mathcal{W}}(e_2) = \sigma_{\mathcal{U}}^{\mathcal{W}}(w_1 - u_1) = \sigma_{\mathcal{U}}^{\mathcal{W}}(w_1) - \sigma_{\mathcal{U}}^{\mathcal{W}}(u_1)$$

$$= -w_1 - u_1 = -\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \underset{-2e_1 - e_2 + 0\cdot e_3}{\underset{\parallel}{\begin{pmatrix}-2\\-1\\0\end{pmatrix}}}$$

$$\sigma_{\mathcal{U}}^{\mathcal{W}}(e_3) = \sigma_{\mathcal{U}}^{\mathcal{W}}(u_2 - w_1 + u_1)$$

$$= \sigma_{\mathcal{U}}^{\mathcal{W}}(u_2) - \sigma_{\mathcal{U}}^{\mathcal{W}}(w_1) + \sigma_{\mathcal{U}}^{\mathcal{W}}(u_1)$$

$$= u_2 + w_1 + u_1$$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix} = 2\cdot e_1 + 2\cdot e_2 + e_3$$

$$M_{\underline{e}}^{\underline{e}}(\sigma_{\mathcal{U}}^{\mathcal{W}}) = \begin{pmatrix}1 & -2 & 2\\ 0 & -1 & 2\\ 0 & 0 & 1\end{pmatrix}$$

Geometricamente