2021-12-22

mercoledì 22 dicembre 2021 08:33

$$\mathbb{R}^3 = U \oplus U^\perp = \left\langle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle \oplus \left\langle \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\1\end{pmatrix} \right\rangle$$

Riflessione di asse $U^\perp$, dove $M^{\mathcal{E}_3}_{\mathcal{E}_3}(\sigma^U_{U^\perp})$

$$e_1 = (e_1 \cdot v_1)\, v_1 + (e_1 \cdot v_2)\, v_2 + (e_1 \cdot w)\, w$$

$$= \frac{1}{\sqrt{2}} \cdot v_1 + \frac{1}{\sqrt{6}} v_2 - \frac{1}{\sqrt{3}} w$$

$$\sigma^U_{U^\perp}(e_1) = -\frac{1}{\sqrt{2}} v_1 - \frac{1}{\sqrt{6}} v_2 - \frac{1}{\sqrt{3}} w$$

$$= -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}-1/3\\-2/3\\-2/3\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

ORA FARE GLI STESSI CONTI CON e_2 e e_3

MATRICE ORTOGONALI $(\mathbb{R})$

→ colore che sono base ortonormale

→ colonne che sono base ortonormale

→ $R^{-1} = R^T \rightsquigarrow RR^T = \mathbb{1} = R^T R$

→ tutti gli autovalori sono ± 1

ISOMETRIE

→ $\|f(v)\| = \|v\| \quad \forall v$

→ la la matrice associata che è ORTOGONALE

$\mathbb{R}^2$ ISOMETRIE:

det $M = 1$ → DIAGONALIZZABILE : identità $\lambda_1 = \lambda_2 = 1$ oppure rotaz π $\lambda_1 = \lambda_2 = -1$

↘ NON DIAG.
* rotazione $\phi \neq 0, \pi$

det $M = -1$ → DIAGONALIZZABILE è una RIFLESSIONE

$(\lambda_1 = 1 \quad \lambda_2 = -1)$

$$* \; M = \begin{pmatrix} \cos\phi & -\text{sen}\,\phi \\ \text{sen}\,\phi & \cos\phi \end{pmatrix}$$

$\mathbb{R}^3$ ISOMETRIA:

$$\left(\begin{array}{c|c} \pm 1 & \\ \hline & \underbrace{B}_{M_{2\times2}} \end{array}\right)$$

$+1$: (DIRETTE)

ROTAZIONE di ASSE v_1 (autovettore associato a $+1$)

$$B = \begin{pmatrix} \cos\theta & -\text{sen}\,\theta \\ \text{sen}\,\theta & \cos\theta \end{pmatrix}$$

-1 : (INVERSE)

ROTAZIONE di asse v_1 ($\uparrow$) + RIFLESSIONE DI ASSE $\langle v_1 \rangle^\perp = \langle v_2, v_3 \rangle$

ESERCIZIO

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \end{pmatrix}$$

Matrice di una isometria rispetto alla base canonica

a) Che isometria è?

$$\det A = 1 \det \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix} =$$

$$= \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$ (DIRETTA

ROTAZIONE DI ASSE v_1 (= autovettore di 1)

$$A - \lambda_i \mathbb{1} = A - \mathbb{1} = \begin{pmatrix} -1 & 0 & 1 \\ 3/5 & -9/5 & 0 \\ 4/5 & 3/5 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 3/5 & -9/5 & 0 & 0 \\ 4/5 & 3/5 & -1 & 0 \end{array}\right) \xrightarrow[\text{III} + \frac{4}{5}\text{I}]{\text{II} + \frac{3}{5}\text{I}} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -9/5 & 3/5 & 0 \\ 0 & 3/5 & -1/5 & 0 \end{array}\right)$$

$$\xrightarrow{II \leftrightarrow III} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 3/5 & -1/5 & 0 \\ 0 & -9/5 & 3/5 & 0 \end{array}\right) \xrightarrow{III - 3II} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} -x_1 + x_3 = 0 \\ \frac{3}{\cancel{5}} x_2 - \frac{1}{\cancel{5}} x_3 = 0 \end{cases} \quad ; \quad x_1 = x_3 \quad ; \quad x_2 = \frac{1}{3} x_3$$

$$V_1 = \left\langle \begin{pmatrix} 1 \\ 1/3 \\ 1 \end{pmatrix} \right\rangle = \left\langle \frac{\cancel{3}}{\sqrt{19}} \cdot \frac{1}{\cancel{3}} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \right\rangle$$

$$= \left\langle \frac{1}{\sqrt{19}} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \right\rangle$$

$$\left(\begin{array}{c|c} 1 & \\ \hline & B \end{array} \right)$$

$$A' = P^T A P$$

matrice di vettori ortonormali con prima colonna v_1

Costruiamo v_2 e v_3 ortonormali a v_1

$$v_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \frac{1}{\sqrt{19}} \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = 0$$

$$\Rightarrow \cancel{\frac{1}{\sqrt{19}}} (3x + y + 3z) = 0$$

$$v_2' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_3 : \quad v_3 \cdot v_1 = 0 \qquad v_3 \cdot v_2 = 0$$

$$3x+y+3z=0 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$x-z=0$$

$$\begin{cases} 3x+y+3z=0 \\ x-z=0 \end{cases} \qquad \begin{cases} 3z+y+3z=0 \\ x=z \end{cases} \qquad \begin{cases} y=-6z \\ x=z \end{cases}$$

$$v_3' = \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix} \qquad v_3 = \frac{1}{\sqrt{38}} \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 3/\sqrt{19} & 1/\sqrt{2} & 1/\sqrt{38} \\ 1/\sqrt{19} & 0 & -6/\sqrt{38} \\ 3/\sqrt{19} & -1/\sqrt{2} & 1/\sqrt{38} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} 3/\sqrt{19} & 1/\sqrt{19} & 3/\sqrt{19} \\ 1/\sqrt{2} & 0 & -1/\sqrt{3} \\ 1/\sqrt{38} & -6/\sqrt{38} & 1/\sqrt{38} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \end{pmatrix} P$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -9/10 & \sqrt{19}/10 \\ 0 & -\sqrt{19}/10 & -9/10 \end{pmatrix} \qquad \begin{cases} \cos\vartheta = -9/10 \\ \operatorname{sen}\vartheta = -\sqrt{19}/10 \end{cases}$$

$$\left(\operatorname{sen}^2\vartheta + \cos^2\vartheta = 1 \right)$$

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Trova P tc $P^T A P = D$ e $P \in O(3, \mathbb{R})$ ORTOGONALE

RISOLVO

$$P_A(x) = \det \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda) \det \begin{pmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2 - 4 \right) =$$

$$= (1-\lambda)\left(1+\lambda^2 - 2\lambda - 4 \right) = (1-\lambda)(\lambda^2 - 2\lambda - 3)$$

$$= (1-\lambda)(\lambda-3)(\lambda+1)$$

$\lambda_1 = -1 \qquad \lambda_2 = 1 \qquad \lambda_3 = 3$

$m_a = m_g = 1 \quad \forall \lambda$

$$A + \mathbb{1} = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \qquad \mathrm{Ker}\,(A + \mathbb{1})$$

$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \xrightarrow{III + I} \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 0 & 2 & 0 & | & 0 \\ -2 & 0 & 2 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{cases} 2x_1 - 2x_3 = 0 \\ x_2 = 0 \end{cases} \qquad \begin{cases} x_1 = x_3 \\ x_2 = 0 \end{cases} \qquad V_{-1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$A - \mathbb{1} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} -2x_3 = 0 \\ -2x_1 = 0 \end{cases} \qquad \begin{cases} x_3 = 0 \\ x_1 = 0 \end{cases} \qquad V_1 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$A - 3\mathbb{1} = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -2 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ -2 & 0 & -2 & | & 0 \end{pmatrix} \xrightarrow{III - I} \begin{pmatrix} -2 & 0 & -2 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{cases} -2x_1 - 2x_3 = 0 \\ -2x_2 = 0 \end{cases} \qquad \begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases} \qquad V_3 = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

RICAVO UNA BASE ORTONORMALE

(SONO ORTOGONALI, E' SUFF. DIVIDERE

(SONO ORTOGONALI, E' SUFF. DIVIDERE PER LA NORMA)

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\mathbb{A}^3(\mathbb{R})$, consideriamo

- il piano M di eq: $x-y-z=1$
- la retta passante per il punto $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ e // al vettore $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (r)
- la retta t di eq: $\begin{cases} x-y=1 \\ z-y=1 \end{cases}$
- il punto $P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

a) Determina eq. cart. e param. di M, r, t

M: $x-y-z=1 \longrightarrow (1 \quad -1 \quad -1 \mid 1)$

eq cartes.

$$a-b-c=1 \Rightarrow a=b+c+1$$

$$\left\{\begin{pmatrix} b+c+1 \\ b \\ c \end{pmatrix}, b,c\in\mathbb{R}\right\} = \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle}_{\text{PTO + DIREZ.}}$$

eq. cartesiane

$$\begin{cases} x=b+c+1 \\ y=b \\ z=c \end{cases}$$

$$t: \begin{cases} x-y=1 \\ z-y=1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\begin{cases} \lambda-\mu=1 \\ -\mu+\rho=1 \end{cases} \qquad \begin{cases} \lambda=1+\mu \\ \rho=1+\mu \end{cases}$$

$$\left\{\begin{pmatrix} 1+\mu \\ \mu \\ 1+\mu \end{pmatrix}, \mu\in\mathbb{R}\right\} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle$$

$$\begin{cases} x=1+\mu \\ y=\mu \\ z=1+\mu \end{cases}$$

$$r: \begin{pmatrix}1\\-1\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle$$

param: (PUNTO + a · vettore)

$$\begin{pmatrix} 1+a \\ -1 \\ +a \end{pmatrix} \qquad \begin{cases} x = 1+a \\ y = -1 \\ z = a \end{cases}$$

eq. cartes. $\longrightarrow$ param eliminando il param.

$$\begin{cases} x = 1+a \\ y = -1 \\ z = a \end{cases} \qquad \begin{cases} x = 1+z \\ y = -1 \\ z = a \end{cases} \qquad \begin{cases} x = 1+z \\ y = -1 \end{cases}$$

$A^3(\mathbb{R})$

$N: \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

$O: \quad x - z = 0$

$N: \quad x - z = 1 \qquad \begin{cases} x = 1 + \overbrace{a+b}^{z} \\ y = -1 + b \\ z = a + b \end{cases}$

$O: \quad \left\{ \begin{pmatrix} 1+\lambda+\mu \\ \mu \\ 1+\lambda+\mu \end{pmatrix}, \ \mu, \lambda \in \mathbb{R} \right\}$

$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$