Foglio di esercizi 1

Esercizio 1:

(a) Si ha $(1+i)^2 = (1+i)(1+i) = 2i$, dunque

$$(1+i)^{10} = ((1+i)^2)^5 = (2i)^5 = 2^5 i^5$$

Poiché $i^2=-1$ si ha $i^5 = i^4 i = (-1)^2 i = i$,

quindi $(1+i)^{10} = 32i$

(b) $z = (1-i)(2+i) = 2+i-2i+1 = 3-i$

semplicemente usando la distributività del prodotto rispetto alla somma e l'identità $i^2=-1$.

(c) $z = 3e^{\frac{5}{6}\pi i} = 3\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right) = \frac{-3\sqrt{3}}{2} + i\frac{3}{2}$

usando la definizione di forma esponenziale e i coseni e seni di angoli notevoli.

(d) $$z = \frac{(\sqrt{2}i+\sqrt{3})^3}{\sqrt{2}-\sqrt{3}i} = \frac{(\sqrt{2}i)^3 + 3(\sqrt{2}i)^2\sqrt{3} + 3(\sqrt{2}i)(\sqrt{3})^2 + (\sqrt{3})^3}{\sqrt{2}-\sqrt{3}i}$$

usando la formula del cubo di un binomio.

Segue

$$z = \frac{-2\sqrt{2}i - 6\sqrt{3} + 9\sqrt{2}i + 3\sqrt{3}}{\sqrt{2}-\sqrt{3}i}$$ semplicemente svolgendo i conti

cioè $$z = \frac{-3\sqrt{3}+7\sqrt{2}i}{\sqrt{2}-\sqrt{3}i} = \frac{(-3\sqrt{3}+7\sqrt{2}i)(\sqrt{2}+\sqrt{3}i)}{(\sqrt{2}-\sqrt{3}i)(\sqrt{2}+\sqrt{3}i)}$$

$$= \frac{-3\sqrt{6}-7\sqrt{6}-9i+14i}{5} = \frac{-10\sqrt{6}+5i}{5} = -2\sqrt{6}+i$$

(e) $z = \dfrac{(1+i)^{10}}{(1-i)^8} = \dfrac{(1+i)^{10}(1+i)^8}{(1-i)^8(1+i)^8} = \dfrac{(1+i)^{18}}{2^8}$

$= \dfrac{((1+i)^2)^9}{2^8} = \dfrac{(2i)^9}{2^8} = \dfrac{2^9}{2^8}\, i^9 = 2\,(i^8)\,i = 2i$

usando $(1+i)^2 = 2i$ e $i^4 = 1$ e razionalizzando.

(f) $z = (i)^{2014} = (i^4)^{503}\, i^2 = i^2 = -1$

usando $2014 = 503 \cdot 4 + 2$, $i^4 = 1$ e $i^2 = -1$.

Esercizio 2:

(a) $z = \sqrt{3} - i$, dunque $\|z\| = \sqrt{3+1} = 2$

segue $z = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$

(b) $z = \sqrt[3]{i-1}$

Poiché $-1+i = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

dalla formula di De Moivre segue che

$z = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + 2k\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + 2k\frac{\pi}{3}\right)\right).$

(c) $z = \left(\dfrac{i-1}{i+1}\right)^3 = \dfrac{\left(\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\right)^3}{\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^3} = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi$

$= -i$

(d) $z = \dfrac{4i}{\sqrt{3}+i} = \dfrac{4i(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \dfrac{4}{4} \; 1+\sqrt{3}i = 1+\sqrt{3}i$

(e) $z = \left(\dfrac{i+\sqrt{3}}{i(\sqrt{3}-i)} \right)^{22} = \left(\dfrac{(i+\sqrt{3})(i+\sqrt{3})}{i(\sqrt{3}-i)(\sqrt{3}+i)} \right)^{22} = \left(\dfrac{2+2\sqrt{3}i}{4i} \right)^{22} =$

$= \dfrac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{22}}{i^{22}} = -\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{22} = -\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{22} =$

$= -\left(\cos\frac{22}{3}\pi + i\sin\frac{22}{3}\pi\right) = -\left(\cos\left(\frac{4}{3}\pi+6\pi\right)+i\sin\left(\frac{4}{3}\pi+6\pi\right)\right)$

$= \frac{1}{2}+\frac{\sqrt{3}}{2}i = \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

(f) $z = e^{-\frac{\pi}{2}i} + e^{-i\frac{\pi}{6}} = -i + \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3}}{2} - \frac{3}{2}i$

$= \sqrt{3}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$

Esercizio 3:

(a) $$z^5 = \frac{\sqrt{3}-i}{\sqrt{3}+i} = \frac{(\sqrt{3}-i)^2}{4} = \frac{2-2\sqrt{3}\,i}{4} = \frac{1}{2}-\frac{\sqrt{3}}{2}i =$$

$$= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$

Per De Moivre si ha

$$z = \cos\left(-\frac{\pi}{15}+\frac{2k\pi}{5}\right) + i\sin\left(-\frac{\pi}{15}+\frac{2k\pi}{5}\right)$$

le radici distinte si ottengono per $k = 0,1,2,3,4$

$$z_0 = \cos\left(-\frac{\pi}{15}\right) + i\sin\left(-\frac{\pi}{15}\right)$$

$$z_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$z_2 = \cos\left(\frac{11}{15}\pi\right) + i\sin\left(\frac{11}{15}\pi\right)$$

$$z_3 = \cos\left(\frac{17}{15}\pi\right) + i\sin\left(\frac{17}{15}\pi\right)$$

$$z_4 = \cos\left(\frac{23}{15}\pi\right) + i\sin\left(\frac{23}{15}\pi\right)$$

(b) $z^3 = \frac{i-1}{i+1} = \frac{(i-1)^2}{2} = \frac{-2i}{2} = -i = \cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)$

Per DeMoivre si ha

$$z = \cos\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{2} + \frac{2k\pi}{3}\right)$$

le radici distinte si ottengono per $k = 0, 1, 2$

$$z_0 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$$z_1 = \cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$z_2 = \cos\left(\frac{11}{6}\pi\right) + i\sin\left(\frac{11}{6}\pi\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

(c) $z^4 = 1$

Per De Moivre si ha

$$z_0 = \cos(0) + i\sin(0) = 1$$

$$z_1 = \cos\left(\frac{2\pi}{4}\right) + i\sin\left(\frac{2\pi}{4}\right) = i$$

$$z_2 = \cos\left(\frac{4\pi}{4}\right) + i\sin\left(\frac{4\pi}{4}\right) = -1$$

$$z_3 = \cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right) = -i$$

(d) $$z^3 = \frac{(i-1)^4}{(i+1)^2} = \frac{\left(\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\right)^4}{\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^2}$$

$$= 2\cos\left(3\pi - \frac{\pi}{2}\right) + i\sin\left(3\pi - \frac{\pi}{2}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

Per De Moivre si ha

$$z = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

le radici distinte si ottengono per $k = 0, 1, 2$

$$z_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt[3]{2}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_1 = \sqrt[3]{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right) = \sqrt[3]{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$z_2 = \sqrt[3]{2}\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right) = -\sqrt[3]{2}\, i$$

(e) $$z^2 = \frac{-2(1-\sqrt{3}i)}{\sqrt{3}-i} = \frac{-2(1-\sqrt{3}i)(\sqrt{3}+i)}{4} = -\sqrt{3} + i$$

$$= 2\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right)$$

Per De Moivre si ha

$$z = \pm\sqrt{2}\left(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi\right)$$

(f) $z^3 = \dfrac{\sqrt{3}-i}{-2+2\sqrt{3}i} = \dfrac{2\left(\cos\frac{-\pi}{6}+i\sin\frac{-\pi}{6}\right)}{4\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)} = \dfrac{1}{2}\left(\cos\frac{-3}{2}\pi+i\sin\frac{-3}{2}\pi\right)$

$$= \frac{1}{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = \frac{1}{2}i$$

Per De Moivre si ha

$$z = \frac{1}{\sqrt[3]{2}}\left(\cos\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{2k\pi}{3}\right)\right)$$

le radici distinte si ottengono per $k=0,1,2$

(g) $z^4 = \dfrac{-2\,(1+\sqrt{3}i)}{\sqrt{3}+i} = \dfrac{-4\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)}{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)} = -2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$$= 2\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

Per De Moivre si ha

$$z = \sqrt[4]{2}\left(\cos\left(\frac{7}{24}\pi+\frac{2k\pi}{4}\right)+i\sin\left(\frac{7}{24}\pi+\frac{2k\pi}{4}\right)\right)$$

le radici distinte si ottengono per $k=0,1,2,3$

(h) $z^5 = \dfrac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)}{4\left(\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right)} = \dfrac{1}{2}\left(\cos\frac{5}{6}\pi+i\sin\frac{5}{6}\pi\right)$

Per De Moivre si ha

$$z = \frac{1}{\sqrt[5]{2}}\left(\cos\left(\frac{\pi}{6}+\frac{2k\pi}{5}\right)+i\sin\left(\frac{\pi}{6}+\frac{2k\pi}{5}\right)\right)$$

le radici distinte si ottengono per $k=0,1,2,3,4$

(i) $$z^2 = \frac{3i}{i+1} = \frac{3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \frac{3}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Per De Moivre si ha

$$z = \pm\frac{\sqrt{3}}{\sqrt[4]{2}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

(l) $$z^3 = \frac{-2i}{i-1} = \frac{2\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right)}{\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)} = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$$

Per De Moivre si ha

$$z = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right)$$

le radici distinte si hanno per $k = 0, 1, 2$

(m) $$z^4 = \frac{3i}{\sqrt{3}i+1} = \frac{3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)} = \frac{3}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Per De Moivre si ha

$$z = \frac{\sqrt[4]{3}}{\sqrt[4]{2}}\left(\cos\left(\frac{\pi}{24} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi}{24} + \frac{2k\pi}{4}\right)\right)$$

le radici distinte si ottengono per $k = 0, 1, 2, 3$

(n) $$z^4 = \frac{5i}{\sqrt{3}-i} = \frac{5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)} = \frac{5}{2}\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$$

Per De Moivre si ha

$$z = \frac{\sqrt[4]{5}}{\sqrt[4]{2}}\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{4}\right)\right)$$

le radici distinte si ottengono per $k = 0, 1, 2, 3$