Foglio di esercizi 8

Esercizio 1:

$$\det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3$$

$$\det\begin{pmatrix} 0 & 1 & 0 & -1 \\ 2 & 3 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} = -\det\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & -2 & 0 \end{pmatrix} - \det\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= -(4-2) - (10+4) = -2-14 = -16$$

$$\det\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 4 & 1 & 2 \end{pmatrix} = -5 + 24 = 19$$

Esercizio 2:

- $\det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} = 1 - 4 + 3 = 0$ la matrice non è invertibile

- $\det\begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix} = 3(7) - 2(-1) + (-5) = 18 \neq 0$

la matrice è invertibile, si ha

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \frac{1}{18} S^t, \quad \text{con } S = \begin{pmatrix} 7 & 1 & -5 \\ -5 & 7 & 1 \\ 1 & -5 & 7 \end{pmatrix}$$

dunque $$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \frac{1}{18}\begin{pmatrix} 7 & -5 & 1 \\ 1 & 7 & -5 \\ -5 & 1 & 7 \end{pmatrix}$$

- $\det\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix} - \det\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} =$

$$= 2 \cdot 2 \cdot 0 - 1 \cdot 0 = 0$$

la matrice non è invertibile

- $\det\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} - \det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$

$$= -\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (-2)\cdot(-2) = 4 \neq 0$$

la matrice è invertibile.

Invertiamola usando la riduzione di Gauss

$$\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \longrightarrow \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{array}$$

$$\xrightarrow[\mathrm{III}+\mathrm{IV}]{\mathrm{I}+\mathrm{II}} \begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{array} \xrightarrow[\frac{1}{2}\mathrm{III}]{\frac{1}{2}\mathrm{I}} \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \end{array}$$

$$\xrightarrow[\mathrm{IV}-\mathrm{III}]{\mathrm{II}-\mathrm{I}} \quad \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1/2 & +1/2 & 0 & 0 \end{array}$$

$$\xrightarrow[-\mathrm{IV}]{-\mathrm{II}} \quad \begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & -1/2 & 0 & 0 \end{array}$$

dunque

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \end{pmatrix}$$

<u>Esercizio 3</u>:

$$\det A_k = \det \begin{pmatrix} 1 & 2 & k \\ k-1 & 1 & 2 \\ 1 & -1 & 2-k \end{pmatrix} = 4-k-2\big((k-1)(2-k)-2\big) - k^2 =$$

$$= 4-k+2k^2-6k+4+4-k^2$$

$$= k^2-7k+12 = (k-3)(k-4)$$

(a) A_k non è iniettivo per $k=3,4$

(b) • $k = 3$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

Risolviamo il sistema che definisce $\ker A_3$:

$$\begin{matrix} 1 & -1 & -1 \\ 1 & 2 & 3 \\ 2 & 1 & 2 \end{matrix} \xrightarrow[\mathrm{III}-2\mathrm{I}]{\mathrm{II}-\mathrm{I}} \begin{matrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{matrix} \xrightarrow{\mathrm{III}-\mathrm{II}} \begin{matrix} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{matrix}$$

il sistema è equivalente al sistema

$$\begin{cases} x - y - z = 0 \\ 3y + 4z = 0 \end{cases}$$

dunque $\ker A_3 = \langle (1, 4, -3) \rangle$

inoltre $\mathrm{Im}\, A_3 = \langle (1,2,1), (2,1,-1) \rangle$

• $k = 4$

$$A_4 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$$

Risolviamo il sistema che definisce $\ker A_4$:

$$\begin{matrix} 1 & -1 & -2 \\ 1 & 2 & 4 \\ 3 & 1 & 2 \end{matrix} \xrightarrow[\mathrm{III}-3\mathrm{I}]{\mathrm{II}-\mathrm{I}} \begin{matrix} 1 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & 4 & 8 \end{matrix} \xrightarrow[\frac{1}{4}\mathrm{III}]{\frac{1}{3}\mathrm{II}} \begin{matrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{matrix}$$

$$\xrightarrow{\mathrm{III}-\mathrm{II}} \begin{matrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$$

il sistema è equivalente al sistema

$$\begin{cases} x - y - 2z = 0 \\ y + 2z = 0 \end{cases}$$

dunque $\ker A_4 = \langle (0, 2, -1) \rangle$

inoltre $\operatorname{Im} A_4 = \langle (1,3,1), (2,1,-1) \rangle$.

(c) A_k iniettivo dunque $k \neq 3, 4$

calcoliamo A_k^{-1}.

$$A_k^{-1} = \frac{1}{\det A_k} S_k^t = \frac{1}{(k-3)(k-4)} S_k^t$$

con

$$S_k = \begin{pmatrix} -k & k^2 - 3k + 4 & -k \\ k-4 & 2-2k & 3 \\ 4-k & k^2 - k - 2 & 3-2k \end{pmatrix}$$

dunque

$$A_k^{-1} = \frac{1}{(k-3)(k-4)} \begin{pmatrix} -k & k-4 & 4-k \\ k^2-3k+4 & 2-2k & k^2-k-2 \\ -k & 3 & 3-2k \end{pmatrix}$$

e quindi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_k^{-1} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{(k-3)(k-4)} \begin{pmatrix} 8-3k \\ k^2+k \\ -k-6 \end{pmatrix}$$

Esercizio 4:

(a)

$$\det\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = 2-6 = -4 \neq 0$$

dunque $\{(1,1,1), (2,1,0), (0,-1,2)\}$ è una base di $\mathbb{R}^3$

$$\det\begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix} = 12-3 = 9 \neq 0$$

dunque $\{(3,1,0), (0,-1,3), (-1,-1,-1)\}$ è una base di $\mathbb{R}^3$.

(b)

Si ha

$$A_{\mathcal{A},\mathcal{B},\mathrm{id}_{\mathbb{R}^3}} = A_{\mathcal{E},\mathcal{B},\mathrm{id}_{\mathbb{R}^3}} \, A_{\mathcal{A},\mathcal{E},\mathrm{id}_{\mathbb{R}^3}} = \left(A_{\mathcal{B},\mathcal{E},\mathrm{id}_{\mathbb{R}^3}}\right)^{-1} A_{\mathcal{A},\mathcal{E},\mathrm{id}_{\mathbb{R}^3}}$$

inoltre $$A_{\mathcal{A},\mathcal{E},\mathrm{id}_{\mathbb{R}^3}} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

e $$A_{\mathcal{B},\mathcal{E},\mathrm{id}_{\mathbb{R}^3}} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}$$

Calcoliamo le inverse di queste due matrici usando il determinante

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \frac{-1}{4} \begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

e

$$\begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}^{-1} = \frac{1}{9} \begin{pmatrix} 4 & -3 & -1 \\ 1 & -3 & 2 \\ 3 & -9 & -3 \end{pmatrix}$$

dunque si ha

$$A_{\mathcal{A},\mathcal{B},id_{\mathbb{R}^3}} = \frac{1}{9} \begin{pmatrix} 4 & -3 & -1 \\ 1 & -3 & 2 \\ 3 & -9 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 0 & 5 & 1 \\ 0 & -1 & 7 \\ -9 & -3 & 3 \end{pmatrix}$$

e

$$A_{\mathcal{B},\mathcal{E},id_{\mathbb{R}^3}} = \frac{1}{4} \begin{pmatrix} -2 & 4 & 2 \\ 3 & -2 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix} =$$

$$= \frac{1}{4} \begin{pmatrix} -2 & 2 & -4 \\ 7 & -1 & 0 \\ 1 & 5 & 0 \end{pmatrix}$$

Esercizio 5:

(a) $\det\begin{pmatrix}1&2&0\\1&1&-1\\1&0&-2\end{pmatrix} = 4 \neq 0$

dunque $(1,1,1), (2,1,0), (0,1,2)$ sono lin. ind.

segue che $\mathbb{R}^3 = U \oplus V$

Sia $\mathcal{A}$ la base $\{(1,1,1), (2,1,0), (0,-1,2)\}$

dell'esercizio 4, si ha

$$A_{\mathcal{A},\mathcal{A},\sigma} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

Inoltre si ha

$$A_{\mathcal{E},\mathcal{E},\sigma} = A_{\mathcal{A},\mathcal{E},id_{\mathbb{R}^3}} \; A_{\mathcal{A},\mathcal{A},\sigma} \; A_{\mathcal{E},\mathcal{A},id_{\mathbb{R}^3}} =$$

$$= A_{\mathcal{A},\mathcal{E},id_{\mathbb{R}^3}} \; A_{\mathcal{A},\mathcal{A},\sigma} \; A^{-1}_{\mathcal{A},\mathcal{E},id_{\mathbb{R}^3}} =$$

$$= \begin{pmatrix}1&2&0\\1&1&-1\\1&0&2\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} \cdot \frac{-1}{4}\begin{pmatrix}2&-4&-2\\-3&2&1\\-1&2&-1\end{pmatrix}$$

$$= \begin{pmatrix}1&2&0\\1&1&1\\1&0&-2\end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix}-2&4&2\\3&-2&-1\\1&-2&1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4&0&0\\2&0&2\\-4&8&0\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\1/2&0&1/2\\-1&2&0\end{pmatrix}$$

Esercizio 6: Sia $A_a = A_{\mathcal{E},\mathcal{E},\phi_a}$ la matrice associata a ϕ_a rispetto alla base canonica $\mathcal{E}$ di $\mathbb{R}^3$.

(a) Si ha

$$A_a = \begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix}$$

(b)

$$\det A_a = \det \begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix} = 2-2a-1-a(-a)$$

$$= a^2-2a+1 = (a-1)^2$$

dunque ϕ_a non è suriettivo per $a=1$

(c)

$$\ker \phi_1 = \ker A_1 = \ker \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \langle (1,-1,0) \rangle$$

inoltre $\operatorname{Im} \phi_1 = \langle (1,0,1), (0,1,2) \rangle$

(d) Si ha

$$A_{\mathcal{A},\mathcal{A},\phi_a} = A_{\mathcal{E},\mathcal{A},id_{\mathbb{R}^3}} \, A_{\mathcal{E},\mathcal{E},\phi_a} \, A_{\mathcal{A},\mathcal{E},id_{\mathbb{R}^3}}$$

Inoltre $A_{\mathcal{E},\mathcal{A},id_{\mathbb{R}^3}} = \left(A_{\mathcal{A},\mathcal{E},id_{\mathbb{R}^3}}\right)^{-1} = \frac{1}{4}\begin{pmatrix}-2 & 4 & 2\\ 3 & -2 & -1\\ 1 & -2 & 1\end{pmatrix}$

Dunque

$$A_{\mathcal{A},\mathcal{A},\phi_a} = \frac{1}{4}\begin{pmatrix}-2 & 4 & 2\\ 3 & -2 & -1\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}1 & a & 0\\ 0 & 1-a & 1\\ a & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 1 & 1 & -1\\ 1 & 0 & 2\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}-2+2a & -6a+6 & 8\\ 3-a & 5a-3 & -4\\ 1+a & 3a-1 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 1 & 1 & -1\\ 1 & 0 & 2\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}-4a+12 & -2a+4 & 6a+10\\ 4a-4 & 3a+3 & -5a-5\\ 4a & 5a+1 & -3a+1\end{pmatrix}$$