Foglio di esercizi 10

Esercizio 1:

(a) Una base ortonormale di U è $\left\{ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \right\}$

(b) Una base di U è $\{ (1,-1,0,0), (0,0,1,-1), (0,1,-1,0) \}$.

Ortonormalizziamo questa base.

$$u_1 = \frac{1}{\sqrt{2}} (1,-1,0,0)$$

$$u_2' = (0,0,1,-1) - \left((0,0,1,-1) \cdot \frac{1}{\sqrt{2}} (1,-1,0,0) \right) \frac{1}{\sqrt{2}} (1,-1,0,0)$$

$$= (0,0,1,-1)$$

$$u_2 = \frac{1}{\sqrt{2}} (0,0,1,-1)$$

$$u_3' = (0,1,-1,0) - \left((0,1,-1,0) \cdot \frac{1}{\sqrt{2}} (1,-1,0,0) \right) \frac{1}{\sqrt{2}} (1,-1,0,0) -$$

$$- \left((0,1,-1,0) \cdot \frac{1}{\sqrt{2}} (0,0,1,-1) \right) \frac{1}{\sqrt{2}} (0,0,1,-1)$$

$$= (0,1,-1,0) + \frac{1}{2} (1,-1,0,0) + \frac{1}{2} (0,0,1,-1) =$$

$$= (0,1,-1,0) + \frac{1}{2} (1,-1,1,-1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \right)$$

$$u_3 = \frac{u_3'}{\| u_3' \|} = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \right)$$

Dunque $\left\{ \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0 \right), \left(0, 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right), \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \right) \right\}$ è una base ortonormale di $U^\perp$.

(c) La proiezione ortogonale di $(4,1,2,1)$ su $U^\perp$ è

$$(4,1,2,1) - \left((4,1,2,1) \bullet \tfrac{1}{2}(1,1,1,1) \right) \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) =$$

$$= (4,1,2,1) - (2,2,2,2) = (2,-1,0,-1)$$

(d) La proiezione ortogonale di $(1,0,0,0)$ su $U^\perp$ è

$$(1,0,0,0) - \left((1,0,0,0) \bullet \tfrac{1}{2}(1,1,1,1) \right) \tfrac{1}{2}(1,1,1,1) =$$

$$= (1,0,0,0) - \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4} \right) = \left(\tfrac{3}{4}, \tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{-1}{4} \right)$$

La proiezione ortogonale di $(0,0,0,1)$ su $U^\perp$ è

$$(0,0,0,1) - \left((0,0,0,1) \bullet \tfrac{1}{2}(1,1,1,1) \right) \tfrac{1}{2}(1,1,1,1) =$$

$$= (0,0,0,1) - \left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4} \right) = \left(\tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{3}{4} \right)$$

Dunque la proiezione ortogonale del sottospazio $\langle (1,0,0,0), (0,0,0,1) \rangle$ su $U^\perp$ è

$$\left\langle \left(\tfrac{3}{4}, \tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{-1}{4} \right), \left(\tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{-1}{4}, \tfrac{3}{4} \right) \right\rangle =$$

$$= \langle (3,-1,-1,-1), (-1,-1,-1,3) \rangle .$$

Esercizio 2:

(a)

$$U_1 = \langle (1,0,1), (2,-1,0) \rangle$$

$U_1^{\perp} : \begin{cases} x + z = 0 \\ 2x - y = 0 \end{cases}$ dunque

$$U_1^{\perp} = \langle (1,2,-1) \rangle$$

Una base ortonormale di $U_1^{\perp}$ è $\left\{ \frac{1}{\sqrt{6}} (1,2,-1) \right\}$.

Dunque si ha

$$p_{U_1}(2,-1,3) = (2,-1,3) - \left((2,-1,3) \cdot \frac{1}{\sqrt{6}} (1,2,-1) \right) \frac{1}{\sqrt{6}} (1,2,-1)$$

$$= (2,-1,3) + \frac{1}{2} (1,2,-1) =$$

$$= \left(\frac{5}{2}, 0, \frac{5}{2} \right)$$

(b) I vettori che hanno stessa proiezione ortogonale di $(2,-1,3)$ su U_1 formano l'insieme

$$p_{U_1}^{-1} \left\{ \left(\frac{5}{2}, 0, \frac{5}{2} \right) \right\} = (2,-1,3) + \ker p_{U_1} =$$

$$= (2,-1,3) + \langle (1,2,-1) \rangle .$$

(c) $$U_a = \langle (1,0,1), (1+a, -a, a-1) \rangle$$

Troviamo l'ortogonale $U_a^\perp$ del sottospazio U_a.

$$U_a^\perp : \begin{cases} x + z = 0 \\ (1+a)x - ay + (a-1)z = 0 \end{cases}$$

da cui

$$\begin{cases} z = -x \\ (1+a)x - ay - (a-1)x = 0 \end{cases}$$

cioè

$$\begin{cases} z = -x \\ 2x - ay = 0 \end{cases}$$

dunque

$$U_a^\perp = \langle (a, 2, -a) \rangle$$

[un'altra maniera di procedere era di calcolare il prodotto vettoriale $(1,0,1) \times (1+a, -a, a-1) = (a, 2, -a)$]

Si ha $p_{U_a}(2,-1,3) = (0,0,0)$ se e solo se

$$(2,-1,3) \in U_a^\perp = \langle (a, 2, -a) \rangle$$

ma ciò non avviene per alcun $a \in \mathbb{R}$

[si poteva anche argomentare che $(2,-1,3) \cdot (1,0,1) = 5 \neq 0$ dunque $(2,-1,3)$ non è mai ortogonale a U_a.

(d) Ortogonalizziamo la base $\{(1,0,1),(1+a,-a,a-1)\}$ di V_a.

$$u_1 = \frac{1}{\sqrt{2}}(1,0,1)$$

$$u_2' = (1+a,-a,a-1) - \left((1+a,-a,a-1)\cdot\frac{1}{\sqrt{2}}(1,0,1)\right)\frac{1}{\sqrt{2}}(1,0,1)$$

$$= (1+a,-a,a-1) - (a,0,a)$$

$$= (1,-a,-1)$$

$$u_2 = \frac{u_2}{\|u_2\|} = \frac{(1,-a,-1)}{\sqrt{a^2+2}}$$

Dunque la proiezione ortogonale $p_{V_a}(2,-1,3)$ di $(2,-1,3)$ su V_a è

$$p_{V_a}(2,-1,3) = \left((2,-1,3)\cdot\frac{1}{\sqrt{2}}(1,0,1)\right)\frac{1}{\sqrt{2}}(1,0,1) + \left((2,-1,3)\cdot\frac{1}{\sqrt{a^2+2}}(1,-a,-1)\right)\frac{1}{\sqrt{a^2+2}}(1,-a,-1)$$

$$= \frac{5}{\sqrt{2}}\cdot\frac{(1,0,1)}{\sqrt{2}} + \frac{a-1}{\sqrt{a^2+2}}\cdot\frac{(1,-a,-1)}{\sqrt{a^2+2}}$$

Si ha $\|p_{V_a}(2,-1,3)\| = \sqrt{\frac{25}{2} + \frac{(a-1)^2}{a^2+2}}$

(per il fatto che $\left\{\frac{(1,0,1)}{\sqrt{2}}, \frac{(1,-a,-1)}{\sqrt{a^2+2}}\right\}$ è una base ortonormale di V_a)

Dunque $\| p_{U_a}(2,-1,3) \| = \sqrt{13}$ se e solo se

$$\frac{25}{2} + \frac{(a-1)^2}{a^2+2} = 13$$

razionalizzando si ottiene

$$25a^2 + 50 + 2a^2 - 4a + 2 = 26a^2 + 52$$

dunque

$$a^2 - 4a = 0 \quad \text{ossia} \quad a(a-4) = 0$$

cioè $\| p_{U_a}(2,-1,3) \| = \sqrt{13}$ per $a = 0$ e $a = 4$

Esercizio 3:

I vettori $(2,0,1,-1)$, $(1,0,1,0)$, $(1,-1,-1,0)$ sono lin. ind. dunque costituiscono una base di U.

(a) Determiniamo una base di $U^\perp$.

$U^\perp$ è definito dal sistema la cui matrice dei coefficienti è

$$\begin{matrix} 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & -1 \end{matrix}$$

Risolviamo il sistema riducendo tale matrice a scala

$$\begin{matrix} 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & -1 \end{matrix} \xrightarrow[\mathrm{III}-2\mathrm{I}]{\mathrm{II}-\mathrm{I}} \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 \end{matrix}$$

dunque si ha

$$U^\perp : \begin{cases} x + z = 0 \\ y + 2z = 0 \\ z + w = 0 \end{cases}$$ la variabile libera è w

ponendo $w = 1$ si ottiene $(1,2,-1,1)$

Dunque $U^\perp = \langle (1,2,-1,1) \rangle$.

Una base ortonormale di $U^\perp$ è $\left\{ \frac{1}{\sqrt{7}}(1,2,-1,1) \right\}$.

(b) Ortonormalizziamo la base

$\{(1,0,1,0),\ (1,-1,-1,0),\ (2,0,1,-1)\}$ di U.

$$u_1 = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2' = (1,-1,-1,0) - \left((1,-1,-1,0)\cdot\frac{1}{\sqrt{2}}(1,0,1,0)\right)\frac{1}{\sqrt{2}}(1,0,1,0)$$

$$= (1,-1,-1,0)$$

[i vettori $(1,0,1,0)$ e $(1,-1,-1,0)$ sono ortogonali; per questo motivo li abbiamo ordinati come primo e secondo vettore della base]

$$u_2 = \frac{1}{\sqrt{3}}(1,-1,-1,0)$$

$$u_3' = (2,0,1,-1) - \left((2,0,1,-1)\cdot\frac{1}{\sqrt{2}}(1,0,1,0)\right)\frac{1}{\sqrt{2}}(1,0,1,0) -$$

$$- \left((2,0,1,-1)\cdot\frac{1}{\sqrt{3}}(1,-1,-1,0)\right)\frac{1}{\sqrt{3}}(1,-1,-1,0)$$

$$= (2,0,1,-1) - \left(\frac{3}{2},0,\frac{3}{2},0\right) - \left(\frac{1}{3},\frac{-1}{3},\frac{-1}{3},0\right)$$

$$= \left(\frac{1}{2},0,\frac{-1}{2},-1\right) - \left(\frac{1}{3},\frac{-1}{3},\frac{-1}{3},0\right) =$$

$$= \left(\frac{1}{6},\frac{1}{3},\frac{-1}{6},-1\right) = \frac{1}{6}(1,2,-1,-6)$$

$$u_3 = \frac{1}{\sqrt{42}}(1,2,-1,-6)$$

Una base ortonormale di U è

$$\left\{ \frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{3}}(1,-1,-1,0), \frac{1}{\sqrt{42}}(1,2,-1,-6) \right\}.$$

(c) L'unione di una base ortonormale di U e di una base ortonormale di $U^{\perp}$ è una base ortonormale di $\mathbb{R}^4$ dato che i vettori sono tutti versori e sono a due a due ortogonali.

(d)

$$p_U(1,0,0,0) = (1,0,0,0) - \left((1,0,0,0) \cdot \frac{1}{\sqrt{7}}(1,2,-1,-1)\right) \cdot \frac{1}{\sqrt{7}}(1,2,-1,-1)$$

$$= (1,0,0,0) - \left(\frac{1}{7}, \frac{2}{7}, \frac{-1}{7}, \frac{-1}{7}\right) =$$

$$= \left(\frac{6}{7}, \frac{-2}{7}, \frac{1}{7}, \frac{1}{7}\right).$$

(e) $$p_U(0,0,0,1) = (0,0,0,1) - \left((0,0,0,1) \cdot \frac{1}{\sqrt{7}}(1,2,-1,-1)\right) \cdot \frac{1}{\sqrt{7}}(1,2,-1,-1) =$$

$$= (0,0,0,1) + \left(\frac{1}{7}, \frac{2}{7}, \frac{-1}{7}, \frac{-1}{7}\right) =$$

$$= \left(\frac{1}{7}, \frac{2}{7}, \frac{-1}{7}, \frac{6}{7}\right)$$

Dunque $p_U(\alpha,0,0,\beta) \in \langle(1,0,0,1)\rangle$

se e solo se $\alpha = \beta$, ossia se e solo se $(\alpha,0,0,\beta) \in \langle(1,0,0,1)\rangle$.

Esercizio 4:

(a) Ortonormalizziamo la base $\{(1,0,1,0),(1,1,0,1)\}$ di $V^\perp$.

$$u_1 = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2' = (1,1,0,1) - \left((1,1,0,1)\cdot \frac{1}{\sqrt{2}}(1,0,1,0)\right)\frac{1}{\sqrt{2}}(1,0,1,0)$$

$$= (1,1,0,1) - \left(\tfrac{1}{2},0,\tfrac{1}{2},0\right) = \left(\tfrac{1}{2},1,-\tfrac{1}{2},1\right)$$

$$= \frac{1}{2}(1,2,-1,2)$$

$$u_2 = \frac{1}{\sqrt{10}}(1,2,-1,2)$$

Si ha

$$p_V(1,4,-3,-2) = (1,4,-3,-2) - \left((1,4,-3,-2)\cdot\frac{1}{\sqrt{2}}(1,0,1,0)\right)\cdot$$

$$\cdot\frac{1}{\sqrt{2}}(1,0,1,0) - \left((1,4,-3,-2)\cdot\frac{1}{\sqrt{10}}(1,2,-1,2)\right)\cdot$$

$$\cdot\frac{1}{\sqrt{10}}(1,2,-1,2)$$

$$= (1,4,-3,-2) + (1,0,1,0) - \frac{8}{10}(1,2,-1,2)$$

$$= (2,4,-2,-2) - \left(\frac{4}{5},\frac{8}{5},-\frac{4}{5},\frac{8}{5}\right) = \left(\frac{6}{5},\frac{12}{5},\frac{-14}{5},\frac{-2}{5}\right)$$

Dunque si conclude che

$$p_V(1,4,-3,-2) = \frac{2}{5}(3,6,-7,-1).$$

(b) Poiché $(1,1,0,1)$ e $(1,0,1,0)$ sono dei generatori di $V^\perp$, si ha che

$$V : \begin{cases} x + y \quad + w = 0 \\ x \quad + z \quad = 0 \end{cases}$$

(c) Si ha

$$p_W(1,4,-3,-2) = (1,4,-3,-2) -$$

$$\left((1,4,-3,-2) \cdot \frac{1}{\sqrt{6}}(2,0,1,1)\right) \frac{1}{\sqrt{6}}(2,0,1,1)$$

$$= (1,4,-3,-2) + \frac{1}{2}(2,0,1,1) =$$

$$= \left(2,4,-\frac{5}{2},-\frac{3}{2}\right).$$