Correzione dell'appello del 4/7/2017 (tema A)

Esercizio 1:

(1) $$w = \frac{1}{(1+2i)^2-1} = \frac{1}{-3+4i-1} = \frac{-1}{4(1-i)}$$

$$= \frac{-(1+i)}{8} = -\frac{1}{8} - i\frac{1}{8}$$

(2) $$|w| = \frac{\sqrt{2}}{8} \qquad \arg(w) = \frac{5}{4}\pi$$

$$w = \frac{\sqrt{2}}{8}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$$

$$z^3 = \frac{\sqrt{2}}{8}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$$

Per deMoivre si ha

$$z_k = \frac{\sqrt[6]{2}}{2}\left(\cos\left(\frac{5}{12}\pi + \frac{2k\pi}{3}\right) + i\sin\left(\frac{5}{12}\pi + \frac{2k\pi}{3}\right)\right)$$

per $k=0,1,2$. Dunque le tre soluzioni sono

$$z_0 = \frac{\sqrt[6]{2}}{2}\left(\cos\left(\frac{5}{12}\pi\right) + i\sin\left(\frac{5}{12}\pi\right)\right)$$

$$z_1 = \frac{\sqrt[6]{2}}{2}\left(\cos\left(\frac{13}{12}\pi\right) + i\sin\left(\frac{13}{12}\pi\right)\right)$$

$$z_2 = \frac{\sqrt[6]{2}}{2}\left(\cos\left(\frac{21}{12}\pi\right) + i\sin\left(\frac{21}{12}\pi\right)\right)$$

Esercizio 2 :

(1) $\boxed{\text{Base di } U : \{(0,1,-1,-1),(1,1,0,-2)\}}$

i vettori $(0,1,-1,-1),(1,1,0,-2)$ sono lin. ind. ,

mentre si ha $(1,-1,2,0) = (1,1,0,-2) - 2(0,1,-1,-1)$

$\boxed{\text{base di } V^{\perp} : \{(1,-1,0,0),(1,0,1,-1)\}}$

$$V^{\perp} : \begin{cases} (x,y,z,w)\cdot(1,1,-1,0) = 0 \\ (x,y,z,w)\cdot(1,1,0,1) = 0 \end{cases}$$

Risolvendo il sistema si ottiene

$$V^{\perp} = \langle (1,-1,0,0),(1,0,1,-1)\rangle$$

(2) $u \in U \iff u = a(0,1,-1,-1) + b(1,1,0,-2) \quad a,b \in \mathbb{R}$

$$u \in V^{\perp} \iff \begin{cases} u\cdot(1,1,-1,0) = 0 \\ u\cdot(1,1,0,1) = 0 \end{cases}$$

quindi segue che

$$u = a(0,1,-1,-1) + b(1,1,0,-2) \in U \cap V^{\perp}$$
$$\Updownarrow$$
$$2a + 2b = 0 \quad \text{ossia} \quad a = -b$$

dunque $U \cap V^{\perp} = \langle (1,0,1,-1)\rangle$

$\boxed{\text{Base di } U \cap V^{\perp} : \{(1,0,1,-1)\}}$

poiché $\dim U = \dim V^{\perp} = 2$ e $\dim U \cap V^{\perp} = 1$

dal teorema del dimensioni segue che

$\dim U + V^{\perp} = 3$

Una base di $U + V^{\perp}$ può essere ottenuta aggiungendo ad una base di $V^{\perp}$ un vettore di U che non appartenga a $V^{\perp}$.

$$\boxed{\text{Base di } U+V^{\perp} : \{(1,-1,0,0),(1,0,1,-1),(0,1,-1,-1)\}}$$

(3) base di V : $\{(1,1,-1,0),(1,1,0,1)\}$

ortonormalizzando questa base si ottiene

$$u_1 = \frac{1}{\sqrt{3}}(1,1,-1,0)$$

$$u_2' = (1,1,0,1) - \frac{2}{3}(1,1,-1,0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1\right)$$

$$u_2 = \frac{u_2'}{\|u_2'\|} = \frac{1}{\sqrt{15}}(1,1,2,3)$$

$$\boxed{\text{Base ortonormale di } V : \left\{\frac{1}{\sqrt{3}}(1,1,-1,0), \frac{1}{\sqrt{15}}(1,1,2,3)\right\}}$$

$\pi_V : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ proiezione ortogonale su V

$$\pi_V(2,1,-2,-3) = \frac{5}{3}(1,1,-1,0) - \frac{10}{15}(1,1,2,3)$$

$$= \frac{5}{3}(1,1,-1,0) - \frac{2}{3}(1,1,2,3)$$

$$= \frac{1}{3}(3,3,-9,-6)$$

dunque

$$\boxed{\pi_V(2,1,-2,-3) = (1,1,-3,-2)}$$

Esercizio 3:

(1) $\phi_k : \mathbb{R}^3 \to \mathbb{R}^3$ è invertibile se e solo se $\det A_k \neq 0$

ossia per $k \neq \pm 1$

Dunque

per $k \neq \pm 1$

$\operatorname{Im} \phi_k = \mathbb{R}^3 = \langle (1,0,0), (0,1,0), (0,0,1) \rangle$

$\ker \phi_k = \{(0,0,0)\}$

per $k = \pm 1$

$\operatorname{Im} \phi_k = \langle (1,0,1), (0,1,0) \rangle$

$\ker \phi_k = \langle (0,0,1) \rangle$

(2) $\phi_k^{-1}(1, 2k, k)$ è l'insieme delle soluzioni

del sistema

$$\begin{aligned} x \qquad + (k^2-1) z &= 1 \\ 2y \qquad &= 2k \\ x \qquad &= k \end{aligned}$$

per $k \neq \pm 1$ ammette un'unica soluzione

$\phi_k^{-1}(1, 2k, k) = \{ (k, k, \frac{-1}{k+1}) \}$ per $k \neq \pm 1$

per $k = -1$ non ammette soluzioni

$\phi_k^{-1}(1, 2k, k) = \emptyset$ per $k = -1$

per $k=1$ ammette infinite soluzioni

$$\Phi_k^{-1}(1,2k,k) = (1,1,0) + \langle (0,0,1) \rangle$$

(3)

$$(1,0,0) = (1,1,1) - (0,1,1)$$

$$(0,1,0) = (0,1,1) + (0,0,-1)$$

$$(0,0,1) = -(0,0,-1)$$

dunque la matrice di passaggio da $\mathcal{E}_3$ a $\mathcal{B}$ è

$$A_{\mathcal{E}_3,\mathcal{B},id_{\mathbb{R}^3}} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Inoltre si ha

$$A_{\mathcal{E}_3,\mathcal{B},\Phi_k} = A_{\mathcal{E}_3,\mathcal{B},id_{\mathbb{R}^3}} A_{\mathcal{E}_3,\mathcal{E}_3,\Phi_k}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & k^2-1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ -1 & 2 & 1-k^2 \\ 1 & 0 & k^2-1 \end{pmatrix}$$

Esercizio 4:

(1) $\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = -1 \neq 0$

dunque $\mathcal{B}$ è una base di $\mathbb{R}^3$.

$$A_{\mathcal{B},\mathcal{B},\psi_h} = \begin{pmatrix} 1 & 0 & 0 \\ 1-h & -1 & h \\ 0 & h & -1 \end{pmatrix}$$

$$B_h = A_{\mathcal{E}_3,\mathcal{E}_3,\psi_h} = A_{\mathcal{B},\mathcal{E}_3,id_{\mathbb{R}^3}} \, A_{\mathcal{B},\mathcal{B},\psi_h} \, A_{\mathcal{E}_3,\mathcal{B},id_{\mathbb{R}^3}}$$

$$B_h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1-h & -1 & h \\ 0 & h & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ h+1 & -h & h-1 \\ -1-h & h & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1-h & h & 0 \\ 0 & 0 & h-1 \\ 0 & -1 & h \end{pmatrix}$$

(2) Polinomio caratteristico:

$$P(t) = \det\left(B_h - tI_3\right) = -\left(t + (1+h)\right)\left(t - (h-1)\right)\left(t-1\right)$$

radici: $1, -1-h, -1+h$

autovalori:

per $h \neq 0, \pm 2$ 1 $\quad m_a(1) = 1$ $-1-h$ $\quad m_a(-1-h) = 1$ $-1+h$ $\quad m_a(-1+h) = 1$	per $h = -2$ 1 $\quad m_a(1) = 2$ -3 $\quad m_a(-3) = 1$
per $h = 0$ 1 $\quad m_a(1) = 1$ -1 $\quad m_a(-1) = 2$	per $h = 2$ 1 $\quad m_a(1) = 2$ -3 $\quad m_a(-3) = 1$

autospazi:

$\boxed{h \neq 0, \pm 2}$

$$V_1 = \ker\left(B_h - I_3\right) = \ker\begin{pmatrix} -2-h & h & 0 \\ 0 & -1 & h-1 \\ 0 & -1 & h-1 \end{pmatrix}$$

$$= \langle \left(h(h-1), (h+2)(h-1), h+2 \right) \rangle$$

$$V_{-1-h} = \ker\left(B_h + (1+h) I_3\right) = \ker\begin{pmatrix} 0 & h & 0 \\ 0 & h+1 & h-1 \\ 0 & -1 & 1+2h \end{pmatrix}$$

$$= \langle (1,0,0) \rangle$$

$$V_{-1+h} = \ker\left(B_h - (h-1) I_3\right) = \ker\begin{pmatrix} -2h & h & 0 \\ 0 & 1-h & h-1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \langle (1,2,2) \rangle$$

$\boxed{h = 0}$

$$B_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$V_{-1} = \ker\left(B_0 + I_3\right) = \ker\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \langle (1,0,0), (0,1,1) \rangle$$

$$V_1 = \ker\left(B_0 - I_3\right) = \ker\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} = \langle (0,1,-1) \rangle$$

$\boxed{k=-2}$

$$B_{-2} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & -3 \\ 0 & -1 & -2 \end{pmatrix}$$

$$V_1 = \ker\left(B_{-2} - I_3\right) = \ker\begin{pmatrix} 0 & -2 & 0 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{pmatrix}$$

$$= \langle (1,0,0) \rangle$$

$$V_{-3} = \ker\left(B_{-2} + 3I_3\right) = \ker\begin{pmatrix} 4 & -2 & 0 \\ 0 & 3 & -3 \\ 0 & 2 & -2 \end{pmatrix}$$

$$= \langle (1,2,2) \rangle$$

$\boxed{k=2}$

$$B_2 = \begin{pmatrix} -3 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$V_1 = \ker\left(B_2 - I_3\right) = \ker\begin{pmatrix} -4 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \langle (1,2,2) \rangle$$

$$V_{-3} = \ker\left(B_2 + 3I_3\right) = \ker\begin{pmatrix} 0 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$$

$$= \langle (1,0,0) \rangle$$

(3) B_h è diagonalizzabile per $h \neq \pm 2$

Sia $\mathcal{V}_h$ una base di autovettori per ψ_h per $h \neq \pm 2$, allora si ha

$$A_{\mathcal{V}_h, \mathcal{V}_h, \psi_h} = A_{\mathcal{E}_3, \mathcal{V}_h, id_{\mathbb{R}^3}} \underbrace{A_{\mathcal{E}_3, \mathcal{E}_3, \psi_h}}_{= B_h} A_{\mathcal{V}_h, \mathcal{E}_3, id_{\mathbb{R}^3}}$$

dunque possiamo scegliere

$$D_h = A_{\mathcal{V}_h, \mathcal{V}_h, \psi_h} \quad \text{e} \quad H_h = A_{\mathcal{V}_h, \mathcal{E}_3, id_{\mathbb{R}^3}}$$

Sia $\mathcal{V}_h$ la seguente base di autovettori per ψ_h

$$\mathcal{V}_h = \begin{cases} \{(1,0,0), (0,1,1), (0,1,-1)\} & \text{per } h = 0 \\ \{(h(h-1), (h+2)(h-1), h+2), (1,0,0), (1,2,2)\} & \text{per } h \neq 0, \pm 2 \end{cases}$$

allora le matrici D_h e H_h che corrispondono a queste scelte sono:

$\boxed{h = 0}$

$$D_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$\boxed{h \neq 0, \pm 2}$

$$D_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1-h & 0 \\ 0 & 0 & -1+h \end{pmatrix}, \quad H_h = \begin{pmatrix} h(h-1) & 1 & 1 \\ (h+2)(h-1) & 0 & 2 \\ h+2 & 0 & 2 \end{pmatrix}$$

Esercizio 5:

(1)

$\pi = P_1 + \langle (P_2 - P_1), (P_3 - P_1) \rangle$

$\pi = (3,1,0) + \langle (-2,0,1), (0,1,1) \rangle$

$\pi : x - 2y + 2z = 1$

(2) $r = P_4 + \langle (P_5 - P_4) \rangle$

$r = (0,-2,3) + \langle (2,0,-1) \rangle$

$r : \begin{cases} y = -2 \\ x + 2z = 6 \end{cases}$

(3) r e π sono parallele

$\mathrm{dist}(r, \pi) = \mathrm{dist}(P_4, \pi) = \dfrac{|9|}{\sqrt{9}} = 3$

(4) i punti che ~~a~~ realizzano la distanza minima con la retta r, in π formano la retta

$s = \pi \cap \pi'$ dove $\pi' = (0,-2,3) + \langle (2,0,-1), (1,-2,2) \rangle$

$\pi' : 2x + 5y + 4z = 2$

dunque $s : \begin{cases} x - 2y + 2z = 1 \\ 2x + 5y + 4z = 2 \end{cases}$, $s = (1,0,0) + \langle (2,0,-1) \rangle$