TUTORATO 1

1) Calcolare le seguenti potenze di i:

a) $i^2 = -1$ per definizione

b) $i^3 = i \cdot i^2 = i \cdot (-1) = -i$;

c) $i^4 = i^2 \cdot i^2 = (-1)\cdot(-1) = 1$;

d) $\frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$;

e) $i^{34} = i^{32} \cdot i^2 = (i^4)^8 \cdot (-1) = (1)^8 \cdot (-1) = -1$;

f) $i^{-7} = (i^7)^{-1} = (i^3 \cdot i^4)^{-1} = (-i \cdot 1)^{-1} = \frac{1}{-i} = -\frac{1}{i} = i$;

2) Semplificare le seguenti espressioni:

a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)$

$= \sqrt{2}-i-i+\sqrt{2}i^2 = \cancel{\sqrt{2}}-2i-\cancel{\sqrt{2}} = -2i$;

b) $(3+i)(3-i)\left(\frac{1}{5}+\frac{1}{10}i\right)$

$= (9-i^2)\left(\frac{1}{5}+\frac{1}{10}i\right) = (9-(-1))\left(\frac{1}{5}+\frac{1}{10}i\right) = 10\left(\frac{1}{5}+\frac{1}{10}i\right) = 2+i$;

c) $\frac{5}{(1-i)(2-i)(3-i)} = \frac{5}{(2-3i+i^2)(3-i)} = \frac{5}{(2-3i-1)(3-i)} = \frac{5}{(1-3i)(3-i)} =$

$= \frac{5}{\cancel{3}-10i-\cancel{3}} = -\frac{1}{2i} = \frac{1}{2}i$;

d) $\overline{\overline{z}+3i} = \overline{\overline{z}}+\overline{3i} = z-3i$;

3) Verificare che $z = 1 \pm i$ soddisfa l'equazione $z^2-2z+2=0$

$z = 1+i$. Sostituisco nell'equazione:

$(1+i)^2-2(1+i)+2=0$

$1+\cancel{2i}+i^2-\cancel{2}-\cancel{2i}+\cancel{2}=0$

$1+i^2=0$

$1-1=0$ ✓ $\quad$ $1+i$ soddisfa l'equazione.

$z = 1-i$. Sostituisco nell'equazione:

$(1-i)^2-2(1-i)+2=0$

$1-\cancel{2i}+i^2-\cancel{2}+\cancel{2i}+\cancel{2}=0$

$1+i^2=0$

$1-1=0$ ✓ $\quad$ $1-i$ soddisfa l'equazione.

4) Calcolare il modulo dei seguenti complessi:

a) $1+i-\dfrac{i}{1-2i}$;

$$\left|1+i-\frac{i}{1-2i}\right| = \left|1+i-\frac{i}{1-2i}\cdot\frac{1+2i}{1+2i}\right| = \left|1+i-\frac{i-2}{1+4}\right| = \left|1+i-\frac{i-2}{5}\right| =$$

$$= \left|\frac{5+5i-i+2}{5}\right| = \frac{1}{5}\left|7+4i\right| = \frac{1}{5}\sqrt{49+16} = \frac{1}{5}\sqrt{65}\ ;$$

b) $(1+i)(1-i)(1+\sqrt{3}i)$;

$$\left|(1+i)(1-i)(1+\sqrt{3}i)\right| = \left|(1-i^2)(1+\sqrt{3}i)\right| = \left|2(1+\sqrt{3}i)\right| = 2\left|1+\sqrt{3}i\right| = 2\sqrt{1+3} = 4\ ;$$

c) $\left(\dfrac{1+i}{1-i}-1\right)^2$;

$$\left|\left(\frac{1+i}{1-i}-1\right)^2\right| = \left|\frac{1+i}{1-i}-1\right|^2 = \left|\frac{1+i-1+i}{1-i}\right|^2 = \left|\frac{2i}{1-i}\right|^2 = \left|\frac{2i}{1-i}\cdot\frac{1+i}{1+i}\right|^2 = \left|\frac{2i-2}{1+1}\right|^2 =$$

$$= \left|i-1\right|^2 = 1+1 = 2\ ;$$

5) Mettere in forma trigonometrica i seguenti complessi:

FORMA TRIGONOMETRICA: $z\in\mathbb{C}$ $\boxed{z = |z|\left(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\right)}$

a) $z=i$;

$|z|=1$; $\vartheta=\mathrm{Arg}(z) \Rightarrow \begin{cases}\cos\vartheta=0\\ \sin\vartheta=1\end{cases} \Rightarrow \vartheta=\dfrac{\pi}{2}$ $\quad(\vartheta\in]-\pi,\pi])$

forma trig.: $z=\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}$;

b) $z=1+i$;

$|z|=\sqrt{1+1}=\sqrt{2}$; $\vartheta=\mathrm{Arg}(z) \Rightarrow \begin{cases}\cos\vartheta=1/\sqrt{2}\\ \sin\vartheta=1/\sqrt{2}\end{cases} \Rightarrow \vartheta=\dfrac{\pi}{4}$

forma trig.: $z=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$;

c) $z=\dfrac{1}{3+3i}$;

$$|z| = \left|\frac{1}{3}\frac{1}{1+i}\right| = \frac{1}{3}\left|\frac{1}{1+i}\right| = \frac{1}{3}\left|\frac{1}{1+i}\cdot\frac{1-i}{1-i}\right| = \frac{1}{3}\left|\frac{1-i}{1+1}\right| = \frac{1}{6}\left|1-i\right| = \frac{1}{6}\sqrt{1+1} = \frac{1}{6}\sqrt{2}$$

$z=\dfrac{1}{6}(1-i)$; $\quad\vartheta=\mathrm{Arg}(z) \Rightarrow \begin{cases}\cos\vartheta=1/\sqrt{2}\\ \sin\vartheta=-1/\sqrt{2}\end{cases} \Rightarrow \vartheta=-\dfrac{\pi}{4}$

forma trig.: $z=\dfrac{1}{6}\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right)$;

d) $z=\frac{4i}{\sqrt{3}+i}$;

$$z=\frac{4i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i}=\frac{4i(\sqrt{3}-i)}{3+1}=\frac{\cancel{4}i(\sqrt{3}-i)}{\cancel{4}}=1+\sqrt{3}i$$

$|z|=\sqrt{1+3}=2$; $\quad \vartheta=Arg(z)\Rightarrow\begin{cases}\cos\vartheta=\frac{1}{2}\\ \sin\vartheta=\frac{\sqrt{3}}{2}\end{cases}\Rightarrow\vartheta=\frac{\pi}{3}$

forma trig.: $z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$;

e) $z=(1+i)(2-2i)$;

$z=2(1+i)(1-i)=2(1+1)=4$; $\qquad |z|=4$;

$\vartheta=Arg(z)\Rightarrow\begin{cases}\cos\vartheta=1\\ \sin\vartheta=0\end{cases}\Rightarrow\vartheta=0$

forma trig.: $z=4(\cos 0+i\sin 0)$;

6) Siano a) $z=\frac{2}{\sqrt{3}-i}+\frac{1}{i}$; $\quad$ b) $z=\frac{1+i}{2-2i}$; Scrivere in forma algebrica, in forma trigonometrica i numeri complessi z^2, z^6, z^{22}:

a) $z=\frac{2}{\sqrt{3}-i}+\frac{1}{i}=\frac{2}{\sqrt{3}-i}\cdot\frac{\sqrt{3}+i}{\sqrt{3}+i}+\frac{1}{i}\cdot\frac{i}{i}=\frac{2(\sqrt{3}+i)}{3+1}+\frac{i}{-1}=\frac{\sqrt{3}}{2}+\frac{i}{2}-i=\frac{\sqrt{3}}{2}-\frac{1}{2}i$;

$|z|=\left|\frac{1}{2}(\sqrt{3}-i)\right|=\frac{1}{2}\sqrt{3+1}=\frac{\cancel{2}}{\cancel{2}}=1$

$\vartheta=Arg(z)\Rightarrow\begin{cases}\cos\vartheta=\sqrt{3}/2\\ \sin\vartheta=-1/2\end{cases}\Rightarrow\vartheta=-\frac{\pi}{6}$

•) $|z^2|=1^2=1$; $\quad Arg(z^2)=2Arg(z)=-\frac{\pi}{3}$;

forma trig.: $z^2=\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)$; $\qquad$ forma alg.: $z^2=\frac{1}{2}-i\frac{\sqrt{3}}{2}$;

••) $|z^6|=1$; $\quad Arg(z^6)=6Arg(z)=-\pi=\pi-2\pi$;

forma trig.: $z^6=\cos(\pi)+i\sin(\pi)$; $\qquad$ forma alg.: $z^6=-1$;

•••) $|z^{22}|=1$; $\quad Arg(z^{22})=22\,Arg(z)=-\frac{11}{3}\pi=\frac{\pi}{3}-4\pi$;

forma trig.: $z^{22}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$; $\qquad$ forma alg.: $z^{22}=\frac{1}{2}+i\frac{\sqrt{3}}{2}$

b) $z=\frac{1+i}{2-2i}=\frac{1}{2}\,\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}=\frac{1}{2}\,\frac{2i}{1+1}=\frac{1}{2}i$

$|z|=\frac{1}{2}$; $\quad Arg(z)=\vartheta\Rightarrow\begin{cases}\cos\vartheta=0\\ \sin\vartheta=1\end{cases}\Rightarrow\vartheta=\frac{\pi}{2}$;

•) $|z^2|=\left(\frac{1}{2}\right)^2=\frac{1}{4}$; $\quad Arg(z^2)=2\,Arg(z)=\pi$;

forma trig.: $z^2=\frac{1}{4}(\cos\pi+i\sin\pi)$; $\qquad$ forma alg.: $z^2=-\frac{1}{4}$

••) $|z^6|=\left(\frac{1}{2}\right)^6=\frac{1}{64}$; $\quad Arg(z^6)=6\,Arg(z)=3\pi=\pi+2\pi$;

forma trig.: $z^6=\frac{1}{64}(\cos\pi+i\sin\pi)$; $\qquad$ forma alg.: $z^6=-\frac{1}{64}$;

..) $|z^{22}| = \left(\frac{1}{2}\right)^{22} = \frac{1}{2^{22}}$; $Arg(z^{22}) = 22\,Arg(z) = 11\pi = \pi + 10\pi$;

forma trig.: $z^{22} = \frac{1}{2^{22}}(\cos\pi + i\sin\pi)$; forma alg.: $z^{22} = -\frac{1}{2^{22}}$;

2) Trovare le radici dei seguenti numeri complessi e disegnarle nel piano di Gauss:

a) $\sqrt[4]{\sqrt{2}}$; b) $\sqrt{1-\sqrt{3}i}$;

a) $z = \sqrt[4]{\sqrt{2}} = (2^{1/2})^{1/4} = 2^{1/8} = \sqrt[8]{2}$

Calcolo le radici ottave di 2. $|2| = 2$, $Arg(2) = 0$.

le 8 radici ottave di 2 hanno argomenti $\vartheta_k = 0 + \frac{2k\pi}{8}$ $k = 0, \dots, 7$ e modulo $\sqrt[8]{2}$

Le 8 radici ottave di 2 sono, quindi:

$z_k = \sqrt[8]{2} \cdot (\cos(\vartheta_k) + i\sin(\vartheta_k))$ $k = 0, \dots, 7$

che nel piano di Gauss sono i vertici di un ottagono regolare inscritto in una circonferenza di raggio $\sqrt[8]{2}$ centrata nell'origine, avente uno dei vertici in $(\sqrt[8]{2}, 0)$.

b) $\sqrt{z} = \sqrt{1-\sqrt{3}i}$; $z = 1-\sqrt{3}i$; $|z| = \sqrt{1+3} = 2$; $Arg(z) = \vartheta \Rightarrow \begin{cases} \cos\vartheta = \frac{1}{2} \\ \sin\vartheta = -\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \vartheta = -\frac{\pi}{3}$;

Siano α_0, α_1 le 2 radici quadrate di z:

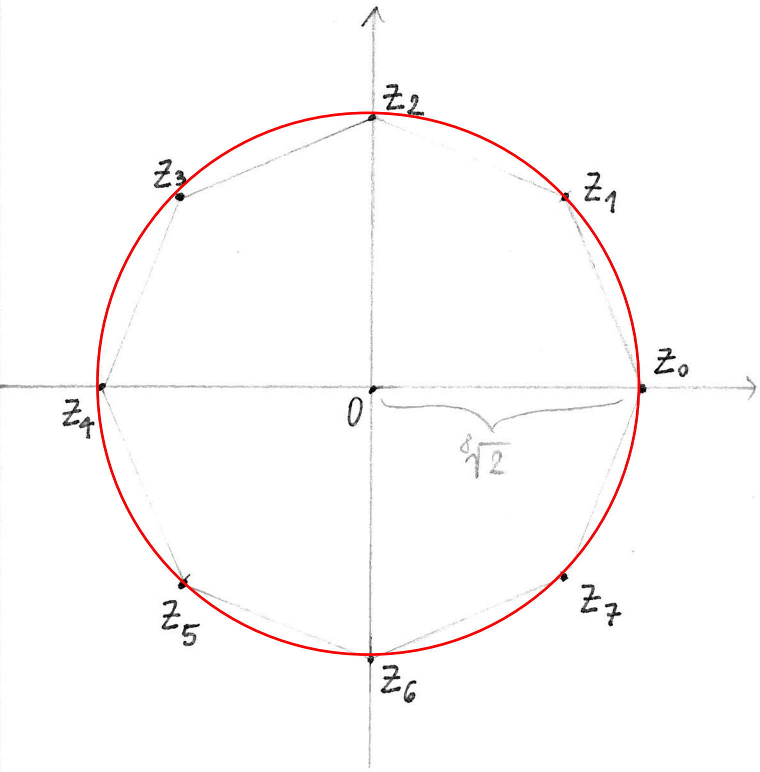
$|\alpha_k| = |\sqrt{z}| = \sqrt{2}$; $Arg(\alpha_k) = \dfrac{-\frac{\pi}{3} + 2k\pi}{2}$ $k = 0, 1$

Quindi:

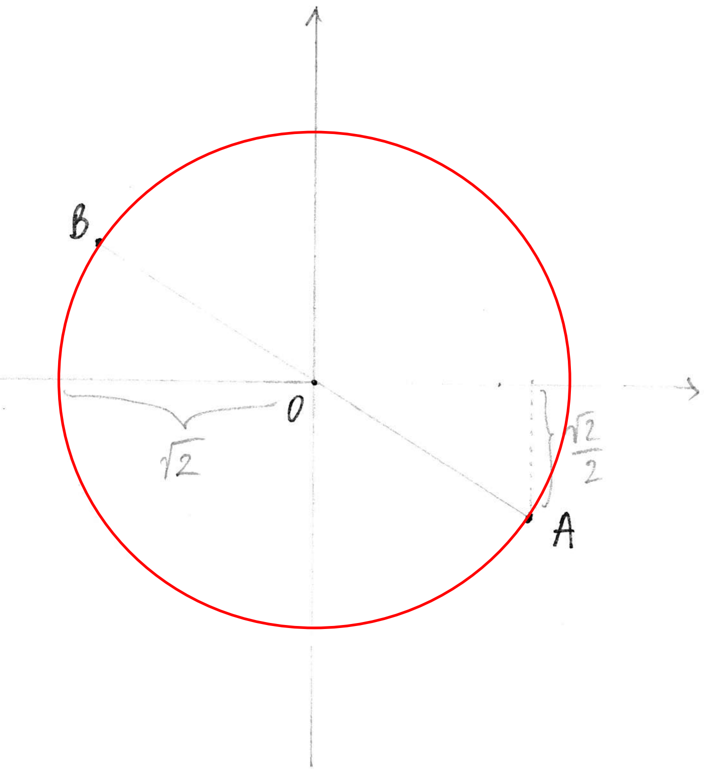
$$\alpha_0 = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$$

$$\alpha_1 = \sqrt{2}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$$

Osservando che $\alpha_0 = -\alpha_1$, nel piano di Gauss sono i punti $A = \left(\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}\right)$, $B = \left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$ simmetrici rispetto all'origine, sulla circonferenza di raggio $\sqrt{2}$ centrata nell'origine.



z_2
z_3
z_1
z_0
z_4
0
$\sqrt[8]{2}$
z_5
z_7
z_6



B
0
$\sqrt{2}$
$\frac{\sqrt{2}}{2}$
A