TUTORATO 2

ES 1

L'insieme dei polinomi di grado $\leq n$ a coefficienti in un campo K è uno spazio vettoriale su K

$P = \{a_0 + a_1 x + \ldots + a_n x^n \mid a_i \in K \ \forall i = 0, \ldots, n\} = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in K \ \forall i = 0, \ldots, n\}$

$+ : P \times P \longrightarrow P, \ (a_0 + a_1 x + \ldots + a_n x^n, b_0 + b_1 x + \ldots + b_n x^n) \longmapsto (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$

$\cdot : K \times P \longrightarrow P, \ (\lambda, a_0 + a_1 x + \ldots + a_n x^n) \longmapsto \lambda a_0 + \lambda a_1 x + \ldots + \lambda a_n x^n$

$a_0 + a_1 x + \ldots + a_n x^n, b_0 + b_1 x + \ldots + b_n x^n, c_0 + c_1 x + \ldots + c_n x^n \in P, \ \lambda, \mu \in K$

1) $[(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)] + (c_0 + c_1 x + \ldots + c_n x^n) =$
$= [(a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n] + (c_0 + c_1 x + \ldots + c_n x^n) =$
$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1) x + \ldots + (a_n + b_n + c_n) x^n =$
$= (a_0 + a_1 x + \ldots + a_n x^n) + [(b_0 + c_0) + (b_1 + c_1) x + \ldots + (b_n + c_n) x^n] =$
$= (a_0 + a_1 x + \ldots + a_n x^n) + [(b_0 + b_1 x + \ldots + b_n x^n) + (c_0 + c_1 x + \ldots + c_n x^n)]$

2) $(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n) = (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n =$
$= (b_0 + a_0) + (b_1 + a_1) x + \ldots + (b_n + a_n) x^n = (b_0 + b_1 x + \ldots + b_n x^n) + (a_0 + a_1 x + \ldots + a_n x^n).$

3) $0_P = 0 + 0x + \ldots + 0x^n$
$(a_0 + a_1 x + \ldots + a_n x^n) + (0 + 0x + \ldots + 0x^n) = (a_0 + 0) + (a_1 + 0) x + \ldots + (a_n + 0) x^n = a_0 + a_1 x + \ldots + a_n x^n.$

4) $-(a_0 + a_1 x + \ldots + a_n x^n) = (-a_0) + (-a_1) x + \ldots + (-a_n) x^n$
$(a_0 + a_1 x + \ldots + a_n x^n) + [-(a_0 + a_1 x + \ldots + a_n x^n)] = (a_0 + a_1 x + \ldots + a_n x^n) + [(-a_0) + (-a_1) x + \ldots + (-a_n) x^n] =$
$= (a_0 - a_0) + (a_1 - a_1) x + \ldots + (a_n - a_n) x^n = 0 + 0x + \ldots + 0x^n = 0_P.$

5) $(\lambda \mu) \cdot (a_0 + a_1 x + \ldots + a_n x^n) = \lambda \mu a_0 + \lambda \mu a_1 x + \ldots + \lambda \mu a_n x^n = \lambda \cdot (\mu a_0 + \mu a_1 x + \ldots + \mu a_n x^n) =$
$= \lambda \cdot [\mu \cdot (a_0 + a_1 x + \ldots + a_n x^n)]$

6) $\lambda \cdot [(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)] = \lambda \cdot [(a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n] =$
$= \lambda (a_0 + b_0) + \lambda (a_1 + b_1) x + \ldots + \lambda (a_n + b_n) x^n = (\lambda a_0 + \lambda b_0) + (\lambda a_1 + \lambda b_1) x + \ldots + (\lambda a_n + \lambda b_n) x^n =$
$= (\lambda a_0 + \lambda a_1 x + \ldots + \lambda a_n x^n) + (\lambda b_0 + \lambda b_1 x + \ldots + \lambda b_n x^n) = \lambda \cdot (a_0 + a_1 x + \ldots + a_n x^n) + \mu \cdot (b_0 + b_1 x + \ldots + b_n x^n)$
$(\lambda + \mu) \cdot (a_0 + a_1 x + \ldots + a_n x^n) = (\lambda + \mu) a_0 + (\lambda + \mu) a_1 x + \ldots + (\lambda + \mu) a_n x^n =$
$= (\lambda a_0 + \mu a_0) + (\lambda a_1 + \mu a_1) x + \ldots + (\lambda a_n + \mu a_n) x^n = (\lambda a_0 + \lambda a_1 x + \ldots + \lambda a_n x^n) + (\mu a_0 + \mu a_1 x + \ldots + \mu a_n x^n) =$
$= \lambda \cdot (a_0 + a_1 x + \ldots + a_n x^n) + \mu \cdot (a_0 + a_1 x + \ldots + a_n x^n).$

7) $1 \cdot (a_0 + a_1 x + \ldots + a_n x^n) = 1a_0 + 1a_1 x + \ldots + 1a_n x^n = a_0 + a_1 x + \ldots + a_n x^n.$

ES 2

Si dica se i seguenti sottoinsiemi di $\mathbb{R}^2$ sono sottospazi vettoriali:

a) $S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = 0\}$; b) $S_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + 4x_2 = 3\}$;
c) $S_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 + x_2^2 = 0\}$

a) $\forall (x_1, x_2), (y_1, y_2) \in S_1, \ \forall \lambda_1, \lambda_2 \in \mathbb{R}$, considero $(z_1, z_2) \overset{!}{=} \lambda_1 (x_1, x_2) + \lambda_2 (y_1, y_2)$
$= (\lambda_1 x_1 + \lambda_2 y_1, \lambda_1 x_2 + \lambda_2 y_2)$

$(x_1, x_2), (y_1, y_2) \in S_1 \Rightarrow 2x_1 - 3x_2 = 0, \ 2y_1 - 3y_2 = 0.$

Devo verificare che:

$(z_1,z_2)\in S_1 \iff 2z_1-3z_2=0 \iff 2(\lambda_1 x_1+\lambda_2 y_1)-3(\lambda_1 x_2+\lambda_2 y_2)=0$

$\iff 2\lambda_1 x_1+2\lambda_2 y_1-3\lambda_1 x_2-3\lambda_2 y_2=0 \iff \lambda_1\underbrace{(2x_1-3x_2)}_{=0}+\lambda_2\underbrace{(2y_1-3y_2)}_{=0}=0$ ✓

$\Rightarrow S_1$ è un sottospazio vettoriale.

b) $\forall (x_1,x_2),(y_1,y_2)\in S_2,\ \forall \lambda_1,\lambda_2\in\mathbb{R},\ (z_1,z_2)=\lambda_1(x_1,x_2)+\lambda_2(y_1,y_2)=(\lambda_1x_1+\lambda_2y_1,\lambda_1x_2+\lambda_2y_2)$

$(x_1,x_2),(y_1,y_2)\in S_2 \iff x_1+4x_2=3,\ y_1+4y_2=3$

Devo verificare che:

$(z_1,z_2)\in S_2 \iff z_1+4z_2=3 \iff (\lambda_1x_1+\lambda_2y_1)+4(\lambda_1x_2+\lambda_2y_2)=3 \iff \lambda_1(x_1+4x_2)+\lambda_2(y_1+4y_2)=3$

$\iff \lambda_1 3+\lambda_2 3=3 \iff (\lambda_1+\lambda_2)3=3 \iff \lambda_1+\lambda_2=1\ \forall\lambda_1,\lambda_2\in\mathbb{R}$ ↯ FALSO.

$\Rightarrow S_2$ NON è un sottospazio vettoriale.

Un altro modo per dimostrare che S_2 non è un sottospazio vettoriale è notare che non contiene lo zero:

$(0,0)\notin S_2$; infatti $0+4\cdot 0=0\neq 3$.

c) $(0,0)\in S_3\ (2\cdot 0+0^2=0)$, $(-2,-2)\in S_3\ (2\cdot(-2)+(-2)^2=-4+4=0)$

$\lambda_1(0,0)+\lambda_2(-2,-2)\in S_3\ \forall\lambda_1,\lambda_2\in\mathbb{R}$

$\lambda_1(0,0)+\lambda_2(-2,-2)=(-2\lambda_2,-2\lambda_2)\in S_3 \iff 2(-2\lambda_2)+(-2\lambda_2)^2=0 \iff -4\lambda_2+4\lambda_2^2=0.$

se $\lambda_2=0$: OK $-4\cdot 0+4\cdot 0=0$ ✓

se $\lambda_2\neq 0$: $-4+4\lambda_2=0 \iff \lambda_2=1\ \forall\lambda_2\in\mathbb{R}$ ↯ FALSO

$\Rightarrow S_3$ NON è un sottospazio vettoriale.

Quindi, ad es, se $\lambda_2=2$
$2(-2,-2)=(-4,-4)\notin S_3$.

ES 3

Si dica se $S=\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid 2x_1-x_2=0,\ x_1+4x_3=1\}$ è un sottospazio vettoriale.

$(0,0,0)\notin S$: $2\cdot 0-0=0$, ma $0+4\cdot 0=0\neq 1 \Rightarrow S$ NON è sottospazio vettoriale.

Oppure, con il ragionamento precedente:

$\forall (x_1,x_2,x_3),(y_1,y_2,y_3)\in\mathbb{R}^3,\ \forall\lambda,\mu\in\mathbb{R}$, considero $(z_1,z_2,z_3)=\lambda(x_1,x_2,x_3)+\mu(y_1,y_2,y_3)$
$=(\lambda x_1+\mu y_1,\lambda x_2+\mu y_2,\lambda x_3+\mu y_3)$

$(x_1,x_2,x_3),(y_1,y_2,y_3)\in S \iff \begin{cases}2x_1-x_2=0\\ x_1+4x_3=1\end{cases},\ \begin{cases}2y_1-y_2=0\\ y_1+4y_3=1\end{cases}$

Devo verificare che:

$(z_1,z_2,z_3)\in S \iff \begin{cases}2z_1-z_2=0\\ z_1+4z_3=1\end{cases} \iff \begin{cases}2(\lambda x_1+\mu y_1)-(\lambda x_2+\mu y_2)=0\\ (\lambda x_1+\mu y_1)+4(\lambda x_3+\mu y_3)=1\end{cases}$

$\iff \begin{cases}\lambda(2x_1-x_2)+\mu(2y_1-y_2)=0\\ \lambda(x_1+4x_3)+\mu(y_1+4y_3)=1\end{cases} \iff \begin{cases}\lambda\cdot 0+\mu\cdot 0=0 \ ✓\\ \lambda\cdot 1+\mu\cdot 1=1 \rightsquigarrow \lambda+\mu=1\ \forall\lambda,\mu\in\mathbb{R}\ ↯\ \text{FALSO.}\end{cases}$

$\Rightarrow S$ NON è un sottospazio vettoriale.

ES 4

Verificare se i seguenti insiemi di vettori di $\mathbb{R}^2$ sono linearmente indipendenti.

a) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$; b) $\left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$; c) $\left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}\right\}$;

[DEF: Un insieme $\{v_1, \dots, v_n\}$ di vettori di uno spazio vettoriale V si dice linearmente indipendente se l'equazione $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$ ha come <u>unica</u> soluzione $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.]

a) $\alpha \begin{pmatrix}1\\0\end{pmatrix} + \beta \begin{pmatrix}1\\1\end{pmatrix} = 0 \iff \begin{cases}\alpha+\beta=0\\ \beta=0\end{cases} \iff \begin{cases}\alpha=0\\ \beta=0\end{cases} \checkmark \Rightarrow \begin{pmatrix}1\\0\end{pmatrix} e \begin{pmatrix}1\\1\end{pmatrix}$ sono lin. ind.

b) $\alpha \begin{pmatrix}2\\3\end{pmatrix} + \beta \begin{pmatrix}1\\1\end{pmatrix} + \gamma \begin{pmatrix}3\\1\end{pmatrix} = 0 \iff \begin{cases}2\alpha+\beta+3\gamma=0\\ 3\alpha+\beta+\gamma=0\end{cases} \underset{I-II}{\iff} \begin{cases}-\alpha+2\gamma=0\\ 3\alpha+\beta+\gamma=0\end{cases} \iff \begin{cases}\alpha=2\gamma\\ 6\gamma+\beta+\gamma=0\end{cases}$

$\iff \begin{cases}\alpha=2\gamma\\ \beta=-7\gamma\end{cases} \Rightarrow \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} e \begin{pmatrix}3\\1\end{pmatrix}$ sono lin. DIPENDENTI.

Infatti, per esempio, se $\gamma = 1 \Rightarrow \alpha = 2, \beta = -7$ e si ha

$2\begin{pmatrix}2\\3\end{pmatrix} - 7\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix} - \begin{pmatrix}7\\7\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-3\\-1\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0$

c) $\alpha \begin{pmatrix}2\\3\end{pmatrix} + \beta \begin{pmatrix}1\\-1\end{pmatrix} + \gamma \begin{pmatrix}3\\2\end{pmatrix} + \delta \begin{pmatrix}1\\1\end{pmatrix} = 0 \iff \begin{cases}2\alpha+\beta+3\gamma+\delta=0\\ 3\alpha-\beta+2\gamma+\delta=0\end{cases} \underset{II-I}{\iff} \begin{cases}2\alpha+\beta+3\gamma+\delta=0\\ \alpha-2\beta-\gamma=0\end{cases}$

$\iff \begin{cases}2\alpha+\beta+3\gamma+\delta=0\\ \alpha=2\beta+\gamma\end{cases} \iff \begin{cases}4\beta+2\gamma+\beta+3\gamma+\delta=0\\ \alpha=2\beta+\gamma\end{cases} \iff \begin{cases}5\beta+5\gamma=-\delta\\ \alpha=2\beta+\gamma\end{cases} \iff \begin{cases}\delta=-5\beta-5\gamma\\ \alpha=2\beta+\gamma\end{cases}$

$\Rightarrow \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ sono lin. dip. Prendendo, infatti, ad es. $\beta = 0, \gamma = 1 \Rightarrow \delta = -5, \alpha = 1$

e si ha $1\begin{pmatrix}2\\3\end{pmatrix} + 0\begin{pmatrix}1\\-1\end{pmatrix} + 1\begin{pmatrix}3\\2\end{pmatrix} - 5\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix} - 5\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix} + \begin{pmatrix}-5\\-5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0$.

d) Basta notare che sono uno multiplo dell'altro: $\begin{pmatrix}3\\6\end{pmatrix} = 3\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow$ sono lin. dipendenti.

Oppure, seguendo il procedimento precedente:

$\alpha \begin{pmatrix}1\\2\end{pmatrix} + \beta \begin{pmatrix}3\\6\end{pmatrix} = 0 \iff \begin{cases}\alpha+3\beta=0\\ 2\alpha+6\beta=0\end{cases} \iff \alpha+3\beta=0 \iff \alpha=-3\beta \Rightarrow \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\6\end{pmatrix}$ sono lin. dip.

Ad es. per $\beta = 1 \Rightarrow \alpha = 3$: $-3\begin{pmatrix}1\\2\end{pmatrix} + 1\begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}-3\\-6\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = 0$.

ES 5

Verificare se i seguenti insiemi di vettori di $\mathbb{R}^3$ sono linearmente indipendenti:

a) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$; b) $\left\{\begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$; c) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}$; d) $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$

a) $\alpha \begin{pmatrix}1\\0\\0\end{pmatrix} + \beta \begin{pmatrix}1\\1\\0\end{pmatrix} + \gamma \begin{pmatrix}1\\2\\1\end{pmatrix} = 0 \iff \begin{cases}\alpha+\beta+\gamma=0\\ \beta+2\gamma=0\\ \gamma=0\end{cases} \iff \begin{cases}\alpha=0\\ \beta=0\\ \gamma=0\end{cases} \Rightarrow$ sono linearmente indipendenti.

b) $\alpha\begin{pmatrix}1\\3\\1\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\0\end{pmatrix}=0 \iff \begin{cases}\alpha+\beta+\gamma=0\\3\alpha+\gamma=0\\\alpha+\beta=0\end{cases} \iff \begin{cases}\gamma=0\\\alpha=0\\\alpha+\beta=0 \rightsquigarrow \beta=0\end{cases}$ $\Rightarrow$ sono lin. indipendenti

c) $\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}+\delta\begin{pmatrix}1\\2\\3\end{pmatrix}=0 \iff \begin{cases}\alpha+\delta=0\\\beta+2\delta=0\\\gamma+3\delta=0\end{cases} \iff \begin{cases}\alpha=-\delta\\\beta=-2\delta\\\gamma=-3\delta\end{cases}$ $\Rightarrow$ sono linearmente DIPENDENTI

Ad es, per $\delta=-1 \Rightarrow \alpha=1,\ \beta=2,\ \gamma=3$: $1\begin{pmatrix}1\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\0\end{pmatrix}+3\begin{pmatrix}0\\0\\1\end{pmatrix}-1\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}-1\\-2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=0.$

d) $\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}+\beta\begin{pmatrix}2\\2\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\1\\0\end{pmatrix}=0 \iff \begin{cases}\alpha+2\beta+\gamma=0\\\alpha+2\beta+\gamma=0\\\alpha+\beta=0\end{cases} \iff \begin{cases}\alpha+2\beta+\gamma=0\\\alpha+\beta=0\end{cases} \overset{\text{I-II}}{\iff} \begin{cases}\beta+\gamma=0\\\alpha+\beta=0\end{cases} \iff \begin{cases}\gamma=-\beta\\\alpha=-\beta\end{cases}$

$\Rightarrow$ linearmente dipendenti. Ad es, per $\beta=-1 \Rightarrow \alpha=1,\ \gamma=1$: $\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}2\\2\\1\end{pmatrix}+\begin{pmatrix}1\\1\\0\end{pmatrix}=0.$

ES 6

Verificare se i seguenti insiemi di matrici di $\mathbb{R}^{2\times 3}$ sono linearmente indipendenti:

a) $\left\{\begin{pmatrix}1&1&0\\2&3&1\end{pmatrix},\begin{pmatrix}0&2&0\\1&0&2\end{pmatrix},\begin{pmatrix}0&0&3\\0&2&1\end{pmatrix}\right\}$; b) $\left\{\begin{pmatrix}1&1&1\\1&2&1\end{pmatrix},\begin{pmatrix}2&1&1\\1&1&1\end{pmatrix}\right\}$.

a) $\alpha\begin{pmatrix}1&1&0\\2&3&1\end{pmatrix}+\beta\begin{pmatrix}0&2&0\\1&0&2\end{pmatrix}+\gamma\begin{pmatrix}0&0&3\\0&2&1\end{pmatrix}=0 \iff \begin{cases}\alpha=0\\\alpha+2\beta=0\\3\gamma=0\\2\alpha+\beta=0\\3\alpha+2\gamma=0\\\alpha+2\beta+\gamma=0\end{cases} \iff \begin{cases}\alpha=0\\\beta=0\\\gamma=0\\0=0\\0=0\\0=0\end{cases} \iff \begin{cases}\alpha=0\\\beta=0\\\gamma=0\end{cases}$

$\Rightarrow$ le 3 matrici sono lin. indipendenti.

b) $\alpha\begin{pmatrix}1&1&1\\1&2&1\end{pmatrix}+\beta\begin{pmatrix}2&1&1\\1&1&1\end{pmatrix}=0 \iff \begin{cases}\alpha+2\beta=0\\\alpha+\beta=0\\\alpha+\beta=0\\\alpha+\beta=0\\2\alpha+\beta=0\\\alpha+\beta=0\end{cases} \iff \begin{cases}\alpha+2\beta=0\\\alpha+\beta=0\\2\alpha+\beta=0\end{cases} \overset{\text{I-II}}{\iff} \begin{cases}\beta=0\\\alpha+\cancel{\beta}=0\\\cancel{2}\alpha=0\end{cases}$

$\iff \begin{cases}\alpha=0\\\beta=0\end{cases}$ $\Rightarrow$ linearmente indipendenti.

ES 7

Consideriamo i seguenti vettori di $\mathbb{R}^4$: $\begin{pmatrix}1\\2\\1\\1\end{pmatrix},\begin{pmatrix}1\\3\\4\\2\end{pmatrix}$. Determinare l'insieme S di tutti i vettori di $\mathbb{R}^4$ che si possono scrivere come combinazione lineare dei 2 vettori dati.

$S=\left\{\alpha\begin{pmatrix}1\\2\\1\\1\end{pmatrix}+\beta\begin{pmatrix}1\\3\\4\\2\end{pmatrix} \,\middle|\, \alpha,\beta\in\mathbb{R}\right\}=\left\{\begin{pmatrix}\alpha+\beta\\2\alpha+3\beta\\\alpha+4\beta\\\alpha+2\beta\end{pmatrix} \,\middle|\, \alpha,\beta\in\mathbb{R}\right\}$

$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in S \iff \begin{cases}x_1=\alpha+\beta\\x_2=2\alpha+3\beta\\x_3=\alpha+4\beta\\x_4=\alpha+2\beta\end{cases} \iff \begin{array}{l}\\ \text{II-2I}\\ \text{III-I}\\ \text{IV-I}\end{array}\begin{cases}x_1=\alpha+\beta\\x_2-2x_1=\beta\\x_3-x_1=3\beta\\x_4-x_1=\beta\end{cases} \iff \begin{cases}x_1=\alpha+x_4-x_1\\x_2-\cancel{2}x_1=x_4-\cancel{x_1}\\x_3-\cancel{x_1}=3x_4-\cancel{3}_{2}x_1\\x_4-x_1=\beta\end{cases} \iff \begin{cases}x_4-2x_1=\alpha\\x_4-x_1=\beta\\2x_1+x_3-3x_4=0\\x_2-x_1-x_4=0\end{cases}$

$\Rightarrow S=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \begin{cases}2x_1+x_3-3x_4=0\\-x_1+x_2-x_4=0\end{cases}\right\}$

ES 8

Stabilire se il seguente insieme di vettori di $\mathbb{R}^3$ è un insieme di generatori di $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix} \right\}.$$

Dato un vettore di $\mathbb{R}^3$ $v = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $x_i \in \mathbb{R}$ per $i = 1,2,3$; verifichiamo che esso si può scrivere nella forma $v = \alpha_1 \begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}1\\2\\1\end{pmatrix} + \alpha_4 \begin{pmatrix}0\\1\\2\end{pmatrix}$ $\alpha_i \in \mathbb{R}$ per $i = 1, \dots, 4$.

$$\Rightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\alpha_1+\alpha_2+\alpha_3\\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4\\ \alpha_2+\alpha_3+2\alpha_4\end{pmatrix} \Leftrightarrow \begin{cases} x_1 = \alpha_1+\alpha_2+\alpha_3 \\ x_2 = \alpha_1+\alpha_2+2\alpha_3+\alpha_4 \\ x_3 = \alpha_2+\alpha_3+2\alpha_4 \end{cases} \Leftrightarrow \begin{cases} x_1 = \alpha_1+\alpha_2+\alpha_3 \\ x_2 - x_1 = \alpha_3+\alpha_4 \\ x_3 = x_2 - x_1 + \alpha_2 + \alpha_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = x_1 - \alpha_2 - \alpha_3 \\ \alpha_3 = x_2 - x_1 - \alpha_4 \\ \alpha_2 = x_1 - x_2 + x_3 - \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \cancel{x_1} - \cancel{x_1} + \cancel{x_2} - x_3 + \alpha_4 - \cancel{x_2} + x_1 + \alpha_4 = x_1 - x_3 + 2\alpha_4 \\ \alpha_2 = x_1 - x_2 + x_3 - \alpha_4 \\ \alpha_3 = x_2 - x_1 - \alpha_4 \end{cases}$$

$\Rightarrow$ è un insieme di generatori, infatti, prendendo ad es $\alpha_4 = 1$: $\begin{cases} \alpha_1 = x_1 - x_3 + 2 \\ \alpha_2 = x_1 - x_2 + x_3 - 1 \\ \alpha_3 = x_2 - x_1 - 1 \end{cases}$

$$\Rightarrow v = (x_1 - x_3 + 2)\begin{pmatrix}1\\1\\0\end{pmatrix} + (x_1 - x_2 + x_3 - 1)\begin{pmatrix}1\\1\\1\end{pmatrix} + (x_2 - x_1 - 1)\begin{pmatrix}1\\2\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\\2\end{pmatrix} =$$

$$= \begin{pmatrix} \cancel{x_1} - \cancel{x_3} + \cancel{2} + x_1 - \cancel{x_2} + \cancel{x_3} - \cancel{1} + \cancel{x_2} - \cancel{x_1} - \cancel{1} \\ \cancel{x_1} - \cancel{x_3} + \cancel{2} + \cancel{x_1} - \cancel{x_2} + \cancel{x_3} - \cancel{1} + \cancel{2}x_2 - 2\cancel{x_1} - \cancel{2} + \cancel{1} \\ \cancel{x_1} - \cancel{x_2} + x_3 - \cancel{1} + \cancel{x_2} - \cancel{x_1} - \cancel{1} + \cancel{2} \end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \checkmark$$

ES 9

Sia V lo spazio vettoriale dei polinomi a coefficienti reali nella variabile x, di grado ≤ 3.
Verificare che $\{1, 1-x, x-x^2, x^2-x^3\}$ è una base di V.

Sia $f(x) \in V$, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, $a_i \in \mathbb{R}$ $i = 0, \dots, 3$

Verifichiamo che si può scrivere come combinazione lineare degli elementi dell'insieme, cioè $f(x) = \lambda_0 \cdot 1 + \lambda_1(1-x) + \lambda_2(x-x^2) + \lambda_3(x^2-x^3)$ $\exists \lambda_i \in \mathbb{R}$ $i = 0, \dots, 3$

$$\Leftrightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \lambda_0 + \lambda_1 - \lambda_1 x + \lambda_2 x - \lambda_2 x^2 + \lambda_3 x^2 - \lambda_3 x^3 =$$
$$= (\lambda_0 + \lambda_1) + (\lambda_2 - \lambda_1)x + (\lambda_3 - \lambda_2)x^2 - \lambda_3 x^3$$

$$\Leftrightarrow \begin{cases} a_0 = \lambda_0 + \lambda_1 \\ a_1 = \lambda_2 - \lambda_1 \\ a_2 = \lambda_3 - \lambda_2 \\ a_3 = -\lambda_3 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = a_0 - \lambda_1 \\ \lambda_1 = \lambda_2 - a_1 \\ \lambda_2 = \lambda_3 - a_2 \\ \lambda_3 = -a_3 \end{cases} \Leftrightarrow \begin{cases} \lambda_0 = a_0 + a_1 + a_2 + a_3 \\ \lambda_1 = -a_1 - a_2 - a_3 \\ \lambda_2 = -a_2 - a_3 \\ \lambda_3 = -a_3 \end{cases}$$

$\Rightarrow$ i polinomi dell'insieme sono dei generatori di V

Se essi sono linearmente indipendenti, allora sono una base di V:

$$\alpha_0 \cdot 1 + \alpha_1(1-x) + \alpha_2 \cdot (x - x^2) + \alpha_3(x^2 - x^3) = 0 \Leftrightarrow (\alpha_0 + \alpha_1) + (\alpha_2 - \alpha_1)x + (\alpha_3 - \alpha_2)x^2 - \alpha_3 x^3 = 0$$

$$\Leftrightarrow \begin{cases} \alpha_0 + \cancel{\alpha_1} = 0 \rightsquigarrow \alpha_0 = 0 \\ \cancel{\alpha_2} - \alpha_1 = 0 \rightsquigarrow \alpha_1 = 0 \\ \cancel{\alpha_3} - \alpha_2 = 0 \rightsquigarrow \alpha_2 = 0 \\ -\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_0 = 0 \\ \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

$\Rightarrow$ sono l. i. $\Rightarrow$ sono una base di V.