

Fundamental Concepts of Synchronous Machine Reactances

By B. R. PRENTICE
ASSOCIATE AIEE

Synopsis

This educational paper presents fundamental concepts of synchronous machine reactances in a form suitable for college seniors and recent graduates. Part I builds up the "physical concepts" of flux linkage and inductance, part II applies the concepts to synchronous machines, developing a physical picture of machine reactances. Part III starts with the fundamental circuit differential equations of a machine and obtains expressions for the reactances in terms of self- and mutual inductances, ending with the machine equations in the direct, quadrature, and zero axes, which give a physical picture of the machine "viewed" from these axes.

Introduction

THIS IS an educational paper designed specifically to assist the college student in bridging the gap between his formal, fundamental education, and current engineering knowledge and practice in the field of synchronous-machine theory. The development of the modern theory of synchronous machines started about 10 years ago when Doherty and Nickle^{1,2,3,4} extended Blondel's 2-reaction theory and developed methods for determining the performance of a salient-pole machine under certain steady-state and transient conditions. They proposed new definitions and divisions of synchronous-machine reactances to facilitate the analysis. Subsequently, Park^{5,6} generalized the method by a more mathematical approach. Wieseman⁷ and Alger⁸ gave methods of calculating the reactances for design, and Park and Robertson⁹ reviewed the whole field of synchronous-machine reactances from both the operating engineer's and designing engineer's viewpoints. Kilgore¹⁰ has given additional methods for calculating the reactances and time constants in design and Wright¹¹ has given methods of test and test results for reactances, resistances, and time constants. Subsequent contributions in the field have been numerous and valuable.

The new ideas presented were and are considerably beyond the training given undergraduates in most colleges. This is as it should be, for there is scarcely time in a college course to do much more than give the student a sound foundation in fundamentals, and this should certainly not be sacrificed in order to progress far in special fields. A

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B. R. PRENTICE is in the engineering general department of the General Electric Company, Schenectady, N. Y.

1. For all numbered references, see list at end of paper.

simple, easily understandable explanation of machine reactances was given by Shildneck in an article,¹² "Synchronous Machine Reactances—A Fundamental and Physical Viewpoint." This paper presents a viewpoint similar to that presented by Shildneck, and a number of others all of which have been found advantageous in the advanced course in engineering¹⁴ of the General Electric Company in the study of synchronous-machine theory.

In the design and analysis of electrical machines, the engineer may or may not need to solve complex mathematical problems so that advanced mathematical knowledge is not essential to every engineer. However, one of his most necessary requirements is a comprehensive, clear understanding of the basic, underlying phenomena. In analysis or design this "physical picture" or "mental picture" is the foundation on which the mathematical relations are set; is the basis for approximations to simplify the formulation and solution; serves as a worthy guide in the straightforward manipulation of the mathematics; and, finally, enables the engineer to interpret his results intelligently. Such a process is present to a considerable degree in the methods of all engineers; those of a highly mathematical bent require a good physical picture to make the fullest use of their powers and those who are endowed with, or have developed, that fine sense that enables them to know the result intuitively—with little or no calculation—have the most complete mental image imaginable.

Since the possession of a clear physical picture is so essential, it is well to investigate how it may be obtained and developed. In mechanics it is easy to visualize forces, torques, velocities, accelerations, mass, etc., because one can see or feel them. The visualization of current, voltage, flux, inductance, etc., is more difficult, and special, artificial concepts such as vectors, sine waves, lines of flux, right-hand rules, etc., are needed to aid the imagination. Since but few of these artifices are either useful or valid in every possible situation, it is desirable to have at one's command a large number of points from which to view a problem. Therefore, the purpose of this paper is to aid in broadening and clarifying the reader's understanding of machine reactances by presenting the concepts from a large number of fundamental points of view.

Part I. Fundamental Concepts of Inductance

Circuit Concepts of Inductance

The basic study of any engineering phenomenon should ordinarily proceed by

1. Gaining or defining fundamental concepts.

2. Observing some experimental relation between the defined concepts.
3. Developing, by means of mathematics, new useful relationships based on the experimental law.

In step 3 it is frequently convenient to define and name additional auxiliary quantities to simplify the analysis and physical reasoning.

The purpose of part I is to develop clear concepts and useful properties of inductance. Proceeding, then, with step 1, primary concepts of the following quantities are presupposed:

- Electromotive force or emf or voltage, e
- Current, i
- Time, t
- Magnetic flux, ϕ
- Flux linkage or (simply) linkage, ψ

The respective units used are: volts, amperes, seconds, lines, lines-turns, the "practical" system.

The linkage of a closed circuit is defined as the sum of the numbers of flux lines linking each turn of the closed circuit. In practice the circuit need not be closed by a wire; if not, the terminals must be remote from the major magnetic field and then the voltage between the terminals is considered to close the circuit. Thus, if in the N turn circuit of figure 1, ϕ_1 lines of flux link turn 1 and ϕ_2 link turn 2, etc., the total linkage is

$$\psi = \phi_1 + \phi_2 + \dots + \phi_N$$

The fundamental, experimentally established law accepted as the basis (in step 2) for this analysis is Faraday's law. It states that whenever the linkage of a closed circuit changes, an electromotive force is induced equal to 10^{-8} times the time rate of decrease of linkage, that is,

$$e = - \frac{d\psi}{dt} 10^{-8} \dagger \quad (1)$$

If the circuit is fixed in the inertial system of the observer, that is if measurements of induced electromotive force are made by, say, attaching a voltmeter to the terminals of the circuit, Faraday's law gives all of the electromotive force induced by changing flux regardless of how the change in linkage is produced, whether by self-induction, relative motion of other circuits, changing current in adjacent circuits, motion or change in permeability of the magnetic circuit, and whether the flux is visualized as moving relative to the coil and cutting it or not. This includes the case where the voltmeter is connected to a moving coil through slip rings for then, electrically, the observer is stationary with respect to the coil. An additional "motional electromotive force" is induced when the circuit is actually moving relative to the observer, as when the voltmeter is attached to a series of similar circuits through a commutator, however it is due entirely to, and is

† Equation 1 establishes an arbitrary convention of sign. Stated differently, a positive applied electromotive force gives rise to a positive increase in linkage, since applied electromotive force is opposite in sense to induced electromotive force. The positive direction of flux and flux linkage is related to the positive sense of electromotive force along the conductor by the right-hand rule. Thus if the fingers of the right hand are pointed in the direction of positive electromotive force along a conductor, the thumb points in the direction of positive flux and flux linkage.

proportional to, the velocity of the circuit relative to the observer's inertial system, and not any real or supposed velocity relative to the field of flux. Page¹⁹ writes, "Indeed the prevalent statement in elementary texts that induced electromotive forces are due to the cutting of lines of force by a conductor is not of general validity and may lead to quite erroneous conclusions in specific problems." In this paper, equations will always be originally set up with the circuits fixed relative to the observer so that Faraday's law gives the total induced electromotive force. This procedure is strongly recommended to avoid errors in circuit problems. Equations involving such a motional electromotive force will be derived in part III, resulting automatically from a transformation to co-ordinates where the observer moves relative to the circuit.

In many practical cases, the linkage of self-induction in a circuit (produced by current in the circuit) is, within reasonable accuracy, directly proportional to the current. When this is the case, the circuit is said to be linear, that is, the linkage is a linear function of the current. Thus

$$\psi = Li \times 10^8 \quad (2)$$

where L is the constant of proportionality. Obviously L is a physical property of the coil and its associated magnetic path. This constant of proportionality occurs so frequently in analysis that it has been named self-inductance. In practical units inductance is then defined by (2) or,

$$L = \frac{\psi}{i} 10^{-8} \text{ henrys} \quad (3)$$

Likewise, the linkage in one circuit is frequently a linear function of the current in another circuit. When this is the case the constant of proportionality is called mutual inductance which is defined by

$$M_{21} = \frac{\psi_{21}}{i_1} 10^{-8} \text{ henrys} \quad (4)$$

The first subscript designates the circuit in which the effect is observed and the second that in which the cause appears. Both self- and mutual inductance can now be visualized as linkage per unit current, or in more usual terms, linkage per ampere.

If the definitions of inductance are combined with equation 1, the following relations result,

$$\left. \begin{aligned} e &= - \frac{d}{dt} (Li) \text{ volts} \\ \text{and} \\ e_2 &= \pm \frac{d}{dt} (M_{21} i_1) \text{ volts} \end{aligned} \right\} \quad (5)$$

* The sign here is arbitrary. Self-inductance is taken as always numerically positive so the positive sign shows that a positive current in a circuit produces positive linkage in the same circuit. The conventions established in equation 1 require that a positive applied electromotive force produce an increase in positive linkage. Hence, according to the convention chosen here, a positive electromotive force will tend to produce a positive current.

** The sign of equation 4 depends on the relative position of the 2 windings and the arbitrary directions chosen for positive current in each circuit. Thus, if positive i_1 produces linkage in circuit 2 in the same direction as produced by positive i_2 , then M is positive. The algebraic sign may be associated with the numerical value of M rather than the literal value as used here.

Even though the inductances are independent of the value of the current, that is constant with respect to i , they may still vary with time, due to the motion of some part of the magnetic circuit or the relative motion of circuits. Hence, it is not permissible to write the more common relations,

$$\left. \begin{aligned} e &= -L \frac{di}{dt} \\ \text{and} \\ e_2 &= \pm M_{21} \frac{di_1}{dt} \end{aligned} \right\} (6)$$

without further restricting the definition to apply to stationary coils with fixed magnetic circuits. This would obviously be folly in the study of rotating machines where circuits move relative to each other and inductances are functions of the position of the rotor, so that equations 5 will be used except when the problem involves only stationary devices.

Inductance in stationary circuits may further be visualized as electromotive-force per unit rate of change of current.

$$L = - \frac{e}{\frac{di}{dt}} \quad (7)$$

Geometric Concept of Inductance

So far, inductance has been considered from the circuit standpoint. However, a clear physical picture of its geometric nature is most important. As defined, inductance is the constant of proportionality between linkage and current. This constant depends upon the size, shape, and permeability of the magnetic path, the number of turns in the coil and their arrangement.

Permeance \mathcal{P} of a magnetic circuit is defined as the ratio of the flux to the magnetomotive force (mmf) producing it. In practical units, the magnetomotive force of a coil is $F = 0.4\pi NI$, so

$$\mathcal{P} = \frac{\phi}{0.4\pi NI} = \frac{N\phi}{0.4\pi N^2 I} = \frac{\psi}{0.4\pi N^2 I} \quad (8)$$

More precisely, if there is fractional linkage as in figure 1, the linkage is not expressible as $N\phi$ so that permeance is defined in terms of linkage as in equation 8. Then inductance, expressed in terms of the geometric property of the circuit, permeance, is

$$L = \frac{\psi}{I} 10^{-8} = 0.4\pi N^2 \mathcal{P} 10^{-8} \quad (9)$$

If the magnetic path can be characterized by single length l , area A , and permeability μ , fundamental consideration of the definition shows that permeance is

$$\mathcal{P} = \frac{\mu A}{l} \quad (10)$$

and in more complex cases, is still a purely geometric con-

stant depending on the magnetic path and arrangement of turns. Thus, inductance is proportional to the number of turns squared, the area of the magnetic path and the permeability, and is inversely proportional to the length of the magnetic path. A similar analysis of the self-inductance of, and mutual inductance between, 2 coils shows that

$$L_1 = 0.4\pi 10^{-8} N_1^2 \mathcal{P}_1 \quad (11)$$

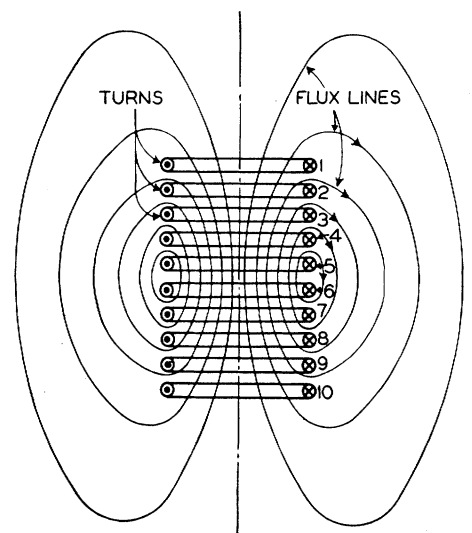
$$L_2 = 0.4\pi 10^{-8} N_2^2 \mathcal{P}_2 \quad (12)$$

$$M_{12} = M_{21} = 0.4\pi 10^{-8} N_1 N_2 \mathcal{P}_{12} \quad (13)$$

where \mathcal{P}_1 and \mathcal{P}_2 represent the respective permeances of the coils and \mathcal{P}_{12} that of the mutual flux path.

The foregoing concepts are extremely useful in visualiz-

Figure 1. Flux linkage



ing the inductances of various windings of a machine, by examining their configuration, their relative arrangement, and the position and character of the magnetic circuit.

Concept of Leakage Inductance

Leakage, prefixed to flux, linkage, inductance, or reactance, is a very broad term being given many interpretations by various authorities. For instance, flux not crossing the useful air gap of an electromagnet is called leakage, primary flux not linking the secondary of a transformer is called leakage, and so on. In general, however, the term is applied to those portions of the quantity so described which are not useful, and on this broad viewpoint are based the various conceptions and definitions used in this paper.

In the simple transformer of figure 2,* let mutual inductance and secondary voltage, linkage, current, and self-inductance be referred to primary terms. That is, if primed quantities are referred to the primary and unprimed quantities are actual values and,

$$n = \frac{N_1}{N_2} = \frac{\text{primary turns}}{\text{secondary turns}}$$

* The transformer of figures 2, 3, and 4 is shown with primary and secondary windings placed on separate legs of the core. Obviously a power transformer would never be constructed in this manner because of the tremendous leakage reactance. However, the arrangement shown is better for the purposes of illustration than the usual concentric location of windings.

then,

$$e_2' = n e_2$$

$$\psi_2' = n \psi_2 \cong \frac{N_1}{N_2} (N_2 \phi_2) = N_1 \phi_2$$

$$i_2' = i_2/n$$

$$L_2' = n^2 L_2 = 0.4\pi 10^{-8} N_1^2 \phi_2$$

$$M' = nM = 0.4\pi 10^{-8} N_1^2 \phi_{12}$$

Since the resistances of transformer or machine windings have little effect on the distribution of flux when either al-

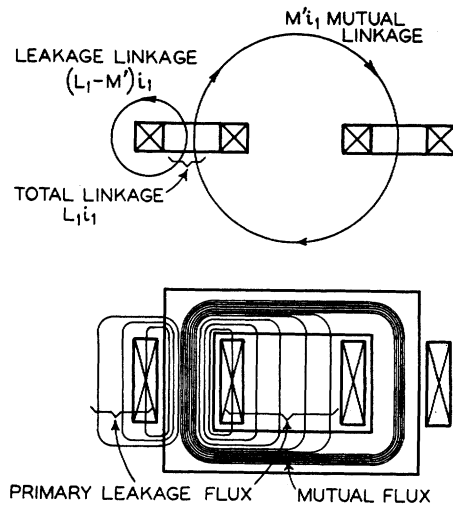


Figure 2. Primary leakage reactance

Flux linkage with primary excited and secondary open

ternating current of rated frequency or rapidly changing direct currents flow, and only these cases are considered here, resistances are neglected.

Now, with the secondary open, apply sinusoidal voltage of angular frequency ω ($\omega = 2\pi f$) to the primary. In complex notation, the voltage equations are

$$\begin{aligned} e_1 &= j\omega\bar{\psi}_1 = j\omega L_1 \bar{i}_1 \dagger \\ e_2' &= j\omega\bar{\psi}_2' = j\omega M' \bar{i}_1 \end{aligned} \quad (14)$$

and the instantaneous primary and secondary linkages are,

$$\begin{aligned} \psi_1 &= L_1 i_1 \\ \psi_2' &= M' i_1 \end{aligned} \quad (15)$$

According to the concept of leakage stated previously, the nonuseful portion of the primary linkage should be defined as the leakage linkage. Obviously the nonuseful portion is that linkage which does not link the secondary or

$$\psi_1 - \psi_2' = (L_1 - M') i_1$$

The various linkages are shown graphically in figure 2.

Therefore, the open-circuit leakage inductance, l_{oc} , of the transformer, referred to the primary is

$$\left. \begin{aligned} l_{oc} &= L_1 - M' \\ \text{and the leakage reactance is} \\ x_{oc} &= \omega (L_1 - M') \end{aligned} \right\} (16)$$

† Bold-face type or overscoring indicates complex or vector representation of the sinusoidal quantities.

If the secondary were excited and primary open, the transformer leakage inductance referred to the primary would be $(L_2' - M')$. These leakage inductances are commonly called primary and secondary leakage inductances, respectively.

It will be evident from what follows that the transformer has an entirely different leakage inductance under every other condition of operation and the purpose of this section is to clarify these diverse concepts of the quantity and build a sound physical picture to be used in visualizing machine reactances.

Consider the transformer operating under the other extreme condition with secondary short circuited and primary excited. Then,

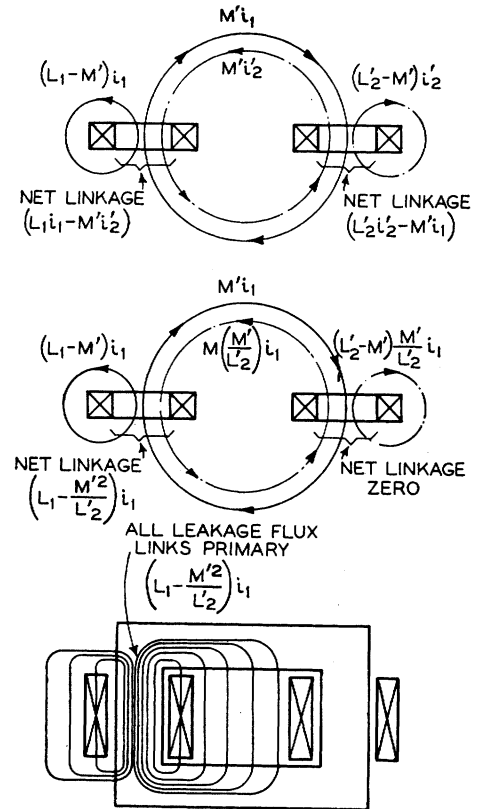
$$\left. \begin{aligned} e_1 &= j\omega\bar{\psi}_1 = j\omega(L_1 \bar{i}_1 - M' \bar{i}_2') \\ 0 &= j\omega\bar{\psi}_2' = j\omega(L_2' \bar{i}_2' - M' \bar{i}_1) \end{aligned} \right\} (17)$$

Positive current conventions are chosen so that M is nega-

Figure 3. Short-circuit reactance

Flux linkage with primary excited with alternating current, secondary short circuited

Full lines show components of linkage produced by primary current, dotted lines, those due to secondary current



tive; that is positive i_1 produces positive ψ_1 , and negative ψ_2 ; positive i_2 produces positive ψ_2 and negative ψ_1 . On elimination of i_2' ,

$$e_1 = j\omega\bar{\psi}_1 = j\omega \left(L_1 - \frac{M'^2}{L_2'} \right) \bar{i}_1 \quad (18)$$

and

$$\psi_1 = \left(L_1 - \frac{M'^2}{L_2'} \right) i_1$$

$$\psi_2' = 0$$

These linkages are shown graphically in figure 3. As

there is no net mutual linkage, or secondary linkage, all of the primary linkage is leakage linkage. Therefore, the transformer short-circuit leakage inductance is

$$l_{sc} = L_1 - \frac{M'^2}{L_2'} \quad \left. \begin{array}{l} \text{and the leakage reactance is} \\ x_{sc} = \omega \left(L_1 - \frac{M'^2}{L_2'} \right) \end{array} \right\} (19)$$

If the secondary were excited with primary short-circuited, the corresponding leakage inductance would be

$$L_2' - \frac{M'^2}{L_1}$$

referred to the primary.

Before comparing these results with (16) it is advantageous to consider one other common definition of leakage reactance, "bucking" reactance. If magnetizing current be neglected, the primary and secondary ampere turns will be equal. This distribution of ampere turns may be accomplished physically by adjusting the voltages applied to primary and secondary such that

$$i_1 = i_2' = i$$

then,

$$e_1 = j\omega\bar{\psi}_1 = j\omega(L_1 i - M' i)$$

$$e_2 = j\omega\bar{\psi}_2' = j\omega(L_2' i - M i)$$

Physical reasoning shows that there is no net mutual linkage. Hence the total linkage is the nonuseful linkage,

$$\psi_1 + \psi_2' = (L_1 + L_2' - 2M')i$$

The relations between the various linkages are shown in figure 4. Hence, the bucking leakage inductance is

$$\begin{aligned} l_B &= L_1 + L_2' - 2M' \\ &= (L_1 - M') + (L_2' - M') \\ &= l_{oc}(\text{primary}) + l_{oc}(\text{secondary}) \end{aligned} \quad (20)$$

and the bucking reactance is

$$x_B = \omega(L_1 + L_2' - 2M')$$

Now to compare the 3 different values of leakage inductance note first that since

$$0.4\pi N_1^2 \Phi_1 10^{-8} = L_1 \cong L_2' = 0.4\pi N_1^2 \Phi_2 10^{-8}$$

the bucking inductance l_B is approximately twice the open-circuit leakage inductance. This may be surprising for inspection of figures 2 and 4 shows that in both cases the leakage flux path in air is of the same area and length. The reason lies in the variation of the magnetomotive force across the leakage path. In figure 2, with the secondary open, the magnetomotive-force drop near the secondary leg of the core is zero and it increases nearly linearly to a maximum at the primary leg of the core. In figure 4, the bucking connection, equal magnetomotive force is supplied by both windings so that the magnetomotive force across the leakage path is essentially constant. With equal primary ampere turns in both cases, the average magnetomotive force in figure 4 is, therefore, approxi-

mately twice the average magnetomotive force in figure 2, and, therefore, so is the leakage inductance.

That the short-circuit inductance and bucking inductance are nearly equal is clear when it is noticed that under short circuit, equation 17 gives

$$i_2' = \frac{M'}{L_2'} i_1$$

and if $M' \cong L_2'$, as in a practical transformer, the primary and secondary ampere turns are nearly equal. Consequently, essentially the same magnetomotive force exists across the same path in figure 3 as in figure 4 and the corresponding leakage inductances l_{sc} and l_B , are essentially the same. The difference is so small it cannot be calculated or tested on commercial power transformers. Subtracting equation 19 from equation 20 gives the difference

$$l_B - l_{sc} = \frac{(L_2' - M')^2}{L_2'}$$

so that the bucking inductance is theoretically slightly larger.

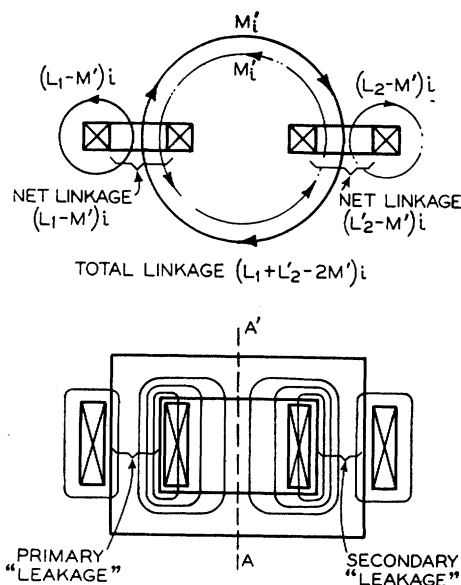
The consequences of this comparison are:

1. The leakage inductance of 2 "closely coupled" circuits varies as much as 2 to 1 depending on operating conditions.
2. Since magnetizing current is usually negligible compared to load or short-circuit current, either l_{sc} or l_B can be used and l_{oc} is of only academic interest.
3. The flux distribution shown in figure 4 for bucking reactance is convenient for the design calculation of reactance but it does not occur physically either under load or short circuit. The device commonly employed for calculation is to assume a surface ($A - A'$ figure 4) dividing the primary leakage from the secondary leakage flux, approximately midway between the 2 windings. Then the reactance associated with the secondary leakage flux is calculated and called secondary leakage reactance and that associated with primary leakage flux, primary leakage reactance. This artifice, though unnecessary, yields satisfactory numerical results. The division between the 2 reactances is purely arbitrary, and they cannot be separated by test.
4. To base a physical picture on the flux distribution of figure 4, as is done in some texts, or to visualize linkage in terms of the transformer equivalent circuit tends to confuse the thinking, particularly in problems involving the law of constant flux linkage.

Figure 4. Bucking reactance

Flux linkage with primary and secondary excited with alternating current such that primary and secondary ampere turns are equal and opposite, $i_1 = i_2' = i$

Full lines show components of linkage due to primary current, dotted lines, those produced by secondary current



5. The flux distribution of figure 3 is the actual condition existing under short circuit (resistance neglected); furthermore, under load the leakage flux links only the excited winding. Therefore, this distribution should be used for purposes of visualization.

A further extension of the concepts of leakage reactance is necessary in order to visualize synchronous-machine transient reactances. Consider the static, short-circuited transformer of figure 3 to have d-c voltage applied suddenly to the primary.

$$\left. \begin{aligned} E1^* &= r_1 i_1 + \frac{d\psi_1}{dt} = r_1 i_1 + \frac{d}{dt}(L_1 i_1 - M' i_2') \\ 0 &= r_2' i_2' + \frac{d\psi_2'}{dt} = r_2' i_2' + \frac{d}{dt}(L_2' i_2' - M' i_1) \end{aligned} \right\} (21)$$

Integrate both equations with respect to time.

$$\begin{aligned} \int_0^t E1^* dt &= \int_0^t r_1 i_1 dt + \int_0^{\psi_1} d\psi_1 \\ 0 &= \int_0^t r_2' i_2' dt + \int_0^{\psi_2'} d\psi_2' \end{aligned}$$

In most practical machines the secondary resistance drop $r_2' i_2'$ is small compared to the induced and applied voltages; hence, if t is made sufficiently small, the first term of the second equation can be made extremely small compared to the terms in the first equation and the change in secondary linkage is practically zero.

$$\psi_2' = \int_0^{\psi_2'} d\psi_2' \cong 0$$

This is one statement of the law of constant flux linkage. Therefore,

$$\psi_2' = L_2' i_2' - M' i_1 \cong 0$$

and

$$i_2' \cong \frac{M'}{L_2'} i_1,$$

and equation 21 becomes

$$E1 \cong r_1 i_1 + \left(L_1 - \frac{M'^2}{L_2'} \right) \frac{di_1}{dt}$$

Hence, exactly initially (t infinitesimally small), and approximately for the first short interval of time, the effective inductance of the short-circuited transformer with suddenly applied d-c voltage is identical with its a-c short-circuit inductance. As time increases indefinitely the effective inductance gradually approaches the self-inductance L_1 .

On first notice it might appear that neglecting the secondary resistance drop in deriving this expression is not warranted when it is intended to use this case as an analogy to machine circuits with the field winding as secondary; for the field resistance is large and cannot be neglected, particularly in determining rates of decay of currents. However, it must be emphasized that the expression is exact when the time interval is infinitesimally small

* The unit function, 1, indicates that the voltage is zero until time zero, E thereafter.

and the effect of high resistance is to shorten the time during which the approximation is a good one.

Physically, this may be explained by realizing that when a sudden voltage is impressed on the primary and the primary current builds up on an exponential type curve, simultaneously the secondary current builds up on an exponential type curve, tending to keep the secondary linkage zero. During the first few instants of time the secondary linkage is essentially zero, and the primary flux is forced entirely into the leakage paths of very low permeance as shown in figure 3.

The constant flux linkage theorem was presented by R. E. Doherty¹⁷ as a tool in the analysis of short-circuit problems. The principles involved are very useful in visualizing and analyzing transient phenomena and it is believed that better use of them could be made in the colleges. Although it will be used, a thorough treatment of the theorem is outside the scope of this paper. It may be briefly stated as follows: the *total linkage* in any closed electrical circuit cannot change instantly. A "closed circuit" is one which involves no infinite voltages or resistances but it may contain any number of finite resistances, inductances (including mutual inductance), and the inductance may vary as a function of time. The rate at which the total linkage does vary is, by Faraday's and Ohm's laws, proportional to the difference between the applied electromotive force and the IR drop.

For the purposes of this paper, the discussion of leakage inductance and reactance serves to build up the following physical picture. When a closed winding is in magnetic proximity to another winding, and the latter winding is excited with either a-c or suddenly applied d-c voltage, the flux linkage of the closed winding remains essentially constant and practically all of the newly created flux present links the excited winding as shown in figure 3. This newly created flux passes through the very low permeance leakage path between the 2 windings so that the effective inductance and reactance are correspondingly small. The initial effective inductance of the excited winding is $(L_1 - M'^2/L_2')$ or approximately $(L_1 + L_2' - 2M')$.

Part II. Synchronous-Machine Reactances— Physical Viewpoint

A fundamental physical picture of each of the more important synchronous-machine reactances will be built up on the basis of the concepts established in part I and considering the effects of a salient-pole rotor and its associated, closed field windings. Laminated cylindrical-rotor machines, being generally easier to visualize, are mentioned only as a special case of the salient-pole machines. Solid cylindrical-rotor machines may be approximately analyzed by salient-pole methods but a more comprehensive theory such as presented by Concordia and Poritsky¹⁸ is necessary for an accurate treatment of transient problems. These concepts are based on the actual phenomena occurring within the machine. It is also possible to view the phenomena from the terminals of the machine, that is, on the basis of actual tests which might

be conducted. Hence, in order to broaden the understanding of these reactances still further, a few methods of determining them by tests—which are simple enough to enhance the viewpoint without resorting to a detailed mathematical analysis—will be included in the discussion of each reactance. Finally, the analytic approach, which gives an understanding of the reactances from still a different angle, will be presented very briefly in part III.

A symmetrical, 3-phase, armature winding is assumed, and saturation, hysteresis, and eddy current losses are neglected. The effect of circuit resistance is neglected in visualizing the linkages but obviously resistance cannot be neglected in visualizing the decay of currents. The field structure and field windings are assumed to be symmetrical about the center lines of the poles, (direct axis) and also symmetrical about the axes midway between the poles (quadrature axis).

Armature Leakage Reactance x_l

The armature leakage reactance of a synchronous machine is a part of nearly all of the other armature reactances, so that a knowledge of it is prerequisite.

If a voltage be applied to the armature terminals such that balanced, sinusoidal, fundamental-frequency, 3-phase currents of normal phase rotation (positive-sequence†) flow in the armature winding, these currents will set up a rotating flux wave in the air gap. Due to the finite number of slots, phases, etc., this flux wave will be composed of fundamental and harmonic components. For instance, the 3-phase currents produce a rotating magnetomotive force which has a fundamental component rotating forward at synchronous speed, a backward rotating fifth harmonic, a forward rotating seventh harmonic, etc., the relative magnitudes of which depend on the winding arrangement. These harmonic magnetomotive forces traveling at speeds different from that of a salient-pole rotor will produce, by interaction with the varying permeance of the air gap, flux harmonics of various orders, traveling at different speeds. (If the rotor is cylindrical and laminated the harmonic magnetomotive forces will give rise to harmonic fluxes of corresponding order and speed of rotation. However, if the cylindrical rotor is solid, currents will be induced in it changing the magnitude of the flux harmonics.) Some of these flux harmonics will be of such order and speed that they will induce voltage of fundamental frequency in the armature. However, they induce purely reactive armature voltage so that their effects are included in the armature leakage reactance, and the corresponding linkage is called differential leakage flux linkage.*

In addition to this armature linkage due to air gap flux

† The usual symmetrical component¹⁶ terms will be used: Positive-sequence (or positive phase-sequence) refers to those symmetrical components of quantities (such as current or voltage) which are of normal (positive) phase rotation; negative-sequence refers to those symmetrical components which are of opposite to normal (negative) phase rotation; zero-sequence refers to those symmetrical components which are of the same time phase or are "in phase" with each other. Any unbalanced 3-phase voltages or currents may be resolved into the 3 sets of symmetrical components listed above. However, an understanding of the system of analysis called symmetrical components is not essential for this paper, it being simply necessary to know the meaning of the defined terms.

* Voltage harmonics of other than fundamental frequency must in all cases be produced by permeance variations or other external influences, and are not themselves causes of reactance.

harmonics, there will be armature linkages due to slot flux and flux in the end winding. If the differential, slot, and end winding linkages are added, the sum is defined to be the total armature leakage flux linkage. The ratio of this linkage in one phase to the armature phase current producing it $\times 10^{-8}$ gives the armature leakage inductance in henries. This value is multiplied by ω , $\omega = 2\pi f$ (where f is rated frequency in cycles per second) is the armature leakage reactance, per phase, in ohms. The ratio of this value to "base ohms" is x_l in per unit.**

It should be noted that the armature leakage reactance includes the effect of that part of the mutual linkage between phases, due to other than the fundamental flux wave and, therefore, is a "3-phase" reactance, applicable without modification only under 3-phase balanced operation or when unbalanced operation is analyzed in terms of its symmetrical components. Unfortunately, this reactance cannot be determined directly from test. However, it is part of nearly all subsequent reactances, and a physical picture of it is, therefore, desirable.

Direct Synchronous Reactance x_d

If positive-sequence currents flow in the armature, as discussed under x_l , the major component of armature linkage is due to the synchronously rotating, fundamental component of the air-gap flux which exists in addition to the leakage flux linkage. Unlike the latter, the amount of linkage due to the fundamental is greatly affected by the position of the field structure. If the unexcited field is rotated synchronously so that the poles are always directly in line with the fundamental component of armature magnetomotive force, maximum permeance is presented to the magnetomotive force as shown in figure 5 and maximum total armature linkage per armature ampere results. Under these conditions the ratio of the linkage in one phase to the current in that phase‡ $\times \omega 10^{-8}$ gives the direct synchronous reactance, per phase,§ in ohms. The ratio of this value to base ohms is x_d in per unit. This reactance obviously includes the armature leakage reactance.

The physical picture of this reactance can be broadened by viewing the phenomena from the terminals of one phase. The phase linkage, due to leakage flux and the synchronously rotating, fundamental, air-gap flux, pulsates sinu-

** The per-unit representation of quantities is similar to the familiar per-cent representation except that the per-unit value of a quantity is the ratio of the actual quantity to an arbitrary amount called the base; whereas the per-cent value is this times 100. Thus, if each phase of a 3-phase machine is rated at 500 amperes and 2,200 volts, "base ohms" is based on the machine rating and is

$$\frac{\text{rated volts}}{\text{rated current}} = \frac{2,200}{500} = 4.4 \text{ ohms}$$

Then if a particular reactance (or resistance or impedance) had an actual value of 8.8 ohms, its per-unit value would be

$$\frac{\text{actual ohms}}{\text{base ohms}} = \frac{8.8}{4.4} = 2 \text{ per unit}$$

If machine reactances are expressed in per unit on the machine rating as base, their numerical values have definite significance to one familiar with machines, and do not vary greatly with the size of machines as do ohmic values. This advantage of the per-unit system will become more apparent when table I is studied.

‡ It has not been stated whether the linkage and current used should be instantaneous, root-mean-square, peak, or some other value. It makes no difference as long as the same kinds are used for both. For indeed, the linkage is a linear function of the current, and their ratio is independent of the kind.

§ Invariably synchronous machine reactances are stated on a per phase base so that further mention of this fact will be omitted.

soidally at normal frequency producing the sinusoidal phase voltage, resistance drop being neglected. Now the axis of the rotating fundamental of armature magnetomotive force will coincide with a phase axis at the instant that that phase current is a maximum. Also, since the field pole is moving synchronously in line with the magnetomotive-force wave, it will be precisely in line with the phase at that instant. Consequently, the phase linkage at that instant will be the maximum possible which can be produced by armature current under any condition. This is the peak value of the sinusoidally varying linkage which induces the sinusoidal phase voltage. As they are proportional, the ratio of the phase voltage to the phase current gives x_d in ohms. It is also apparent that x_d is the largest possible 3-phase armature reactance definable.

In practice it is difficult to rotate the field structure mechanically so that the direct axis is always exactly in line with the axis of the rotating armature magnetomotive force. Consequently such a method is unsuited for the experimental determination of the reactance by test. However, when a synchronous machine is short circuited it is a well-known fact that the current which flows is almost entirely reactive, and, therefore, the armature magnetomotive force is almost directly in line with the poles, tending to demagnetize them. Since this condition obtains automatically it is suitable for determining x_d .

Saturation is neglected so the flux linkage in any one circuit at any particular instant of time is a *linear* function of the magnitudes of the currents existing in every other winding in the machine at that instant and the principle of superposition may then be used to determine the linkage. Thus, the linkage in one circuit due to the currents in certain circuits flowing alone may be determined; that linkage due to the rest of the currents may be determined separately; and then the 2 values of linkage thus obtained when added give the total linkage in the one circuit with all currents flowing. This is true provided that the motion of the rotor is the same in both cases.

Consider the conditions which must exist under steady-state, symmetrical, short circuit. The terminal voltage is zero. Neglecting resistance drop, the net armature linkage including leakage flux linkage, must, therefore, also be zero. Since the net armature linkage is produced by the armature currents and field current acting at the same time, the armature magnetomotive force opposes the field magnetomotive force; the armature linkage due to all armature currents acting alone must be exactly equal to, and opposite in direction to, the armature linkage due to the field current acting alone. If the field current were acting alone (open circuit) a certain terminal voltage would exist which would be proportional to the armature linkage produced by the field current. Now suppose that the armature currents flow alone (field open circuited). Since the armature linkage has the same value, but is in the opposite direction, the voltage which must have been applied to produce the armature currents must be exactly the same as the terminal voltage induced under open circuit. But notice, when the armature currents flow alone the case is exactly that used to define x_d . That is, they are positive phase-sequence, the magnetomotive force

produced by them is in line with the poles, and the field is open. Hence, by definition, the ratio of the terminal phase voltage, which would produce the armature currents alone, to the armature phase current gives x_d in ohms, but this voltage is the same as the open circuit voltage before short circuit. Finally, then, x_d is the ratio of the open-circuit terminal voltage resulting from a certain field current to the steady-state, short-circuit, armature current resulting from the same field current.

$$x_d = \frac{E_{\text{open circuit}}}{I_{\text{short circuit}}} \text{ ohms}$$

Obviously these quantities can be determined from the saturation curve and synchronous impedance curve of the machine.*

This discussion of a possible test to determine x_d has served 2 purposes. First to broaden the physical picture and second to indicate how the principle of superposition may be applied in the solution of synchronous machine problems. For standard definitions and methods of measuring this and subsequent reactances, see the AIEE Standards.²¹

Quadrature Synchronous Reactance x_q

With positive-sequence, steady-state currents flowing in the armature and the unexcited field structure rotated mechanically at synchronous speed, such that the interpolar spaces (quadrature axis) are directly in line with the peaks of the rotating armature magnetomotive-force wave, minimum permeance is presented, and the armature linkage will be the minimum for any position of the poles. The flux paths may be visualized by inspecting figure 6 which shows graphically that the permeance of the air gap in the quadrature axis is much lower than that in the direct axis. (If the rotor is cylindrical the permeances will be equal and $x_d = x_q$.) Under this condition the armature linkage per ampere $\times \omega 10^{-8}$ gives the quadrature synchronous reactance, x_q , in ohms. The ratio of this value to base ohms gives x_q in per unit. This reactance includes the leakage reactance x_l .

If viewed from the armature terminals, the magnitude of the root-mean-square, reactive voltage induced in a phase will be a minimum when the interpolar space is in line with the magnetic axis of the phase at the instant when the current in the phase is a maximum, for if the field structure had any other position at the instant that the phase current was a maximum, a greater armature linkage would result. The ratio of the phase voltage to the phase current under this condition gives x_q in ohms. It is important to note that both x_d and x_q are defined with the field structure in a symmetrical position with respect to the rotating magnetomotive-force wave. Thus, in either case, the field structure has no tendency to be turned by the flux in either direction so that there is no magnetic torque. Therefore, since no electrical power is transferred to the

* The saturation curve is a plot of terminal voltage versus field current and the synchronous impedance curve is a plot of steady-state, short-circuit current versus field current. Normally, the ratio is taken at rated terminal voltage, and to obtain an unsaturated value for x_d the field current at rated terminal voltage on the "air-gap line" is used. The "air-gap line" is a straight line tangent to the straight portion of the magnetization curve.

rotor, the currents are purely reactive (neglecting resistance) and it is legitimate to define a reactance for each case as the ratio of total voltage to total current.

It is difficult to rotate the field structure mechanically so that the quadrature axis is always exactly in line with the armature magnetomotive force for if it deviates slightly a torque will appear (reluctance torque) tending to pull the direct axis into line with the armature magnetomotive force. Hence it is even more impractical in this case to rotate the field structure in its proper position than it was in the case of direct synchronous reactance. However, suppose that the unexcited field is rotated mechanically in the direction of the synchronously revolving armature magnetomotive force, but at a speed slightly below synchronism, and at the same time positive phase-sequence

voltages are impressed at the terminals. Obviously the current will be modulated at slip frequency, being a maximum when the quadrature axis is in line with the armature magnetomotive force and a minimum when the direct axis is in that position. Also, due to the reactance in the supply line, the applied voltage will probably be modulated at slip frequency being a maximum at minimum current. Figure 11 is an oscillogram showing this variation. If the applied voltage and current in one phase are observed by means of instruments and the slip is small enough, the maximum and minimum values of the modulated current and voltage can be read directly from instruments. Minimum voltage and maximum current will occur when the quadrature axis is in line with the magnetomotive-force wave peak. Quadrature synchronous reactance in ohms

Figure 5. Direct synchronous reactance flux paths

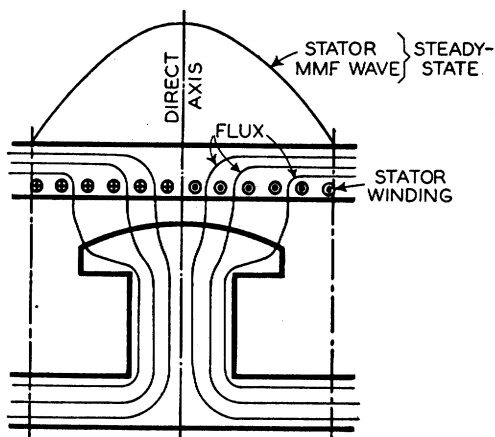


Figure 6. Quadrature synchronous reactance flux paths

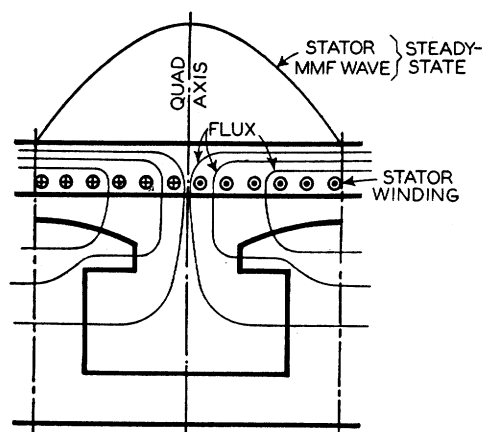


Figure 7. Direct transient reactance flux paths

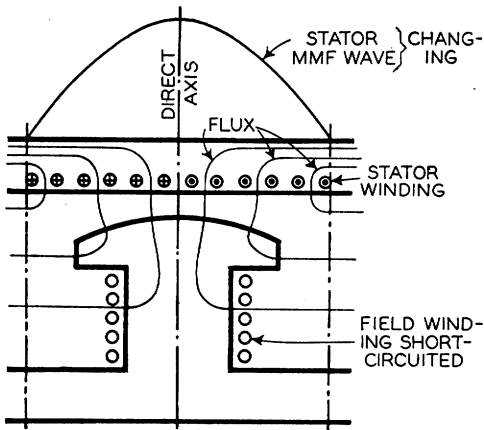


Figure 8. Quadrature transient reactance flux paths

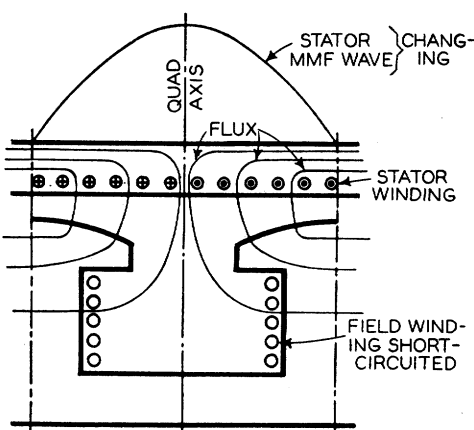


Figure 9. Direct subtransient reactance flux paths

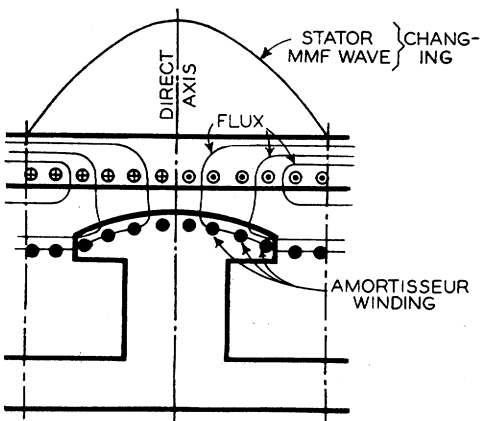
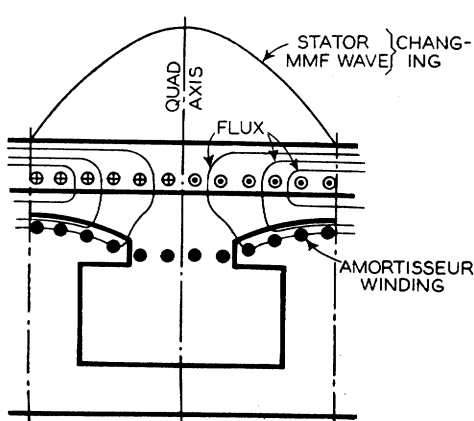


Figure 10. Quadrature subtransient reactance flux paths



is then the ratio of minimum voltage to maximum current. From this same test, the ratio of maximum voltage to minimum current gives direct synchronous reactance in ohms.

Direct Transient Reactance x_d'

Direct synchronous reactance is analogous to the self-inductance of the primary of the open-circuit transformer of figure 2. This is true regardless of whether the rotor

rotor circuits, such as the amortisseur or damper windings, imagined to be open. Suddenly, terminal voltages are applied at such an instant that fundamental-frequency, positive-sequence armature currents result* and produce a rotating magnetomotive force the axis of which is directly in line with the poles. This condition is identical with that used for defining direct synchronous reactance except that the voltages are now suddenly applied. The sudden appearance of the magnetomotive force in the direct axis is resisted by the current induced in the closed field winding

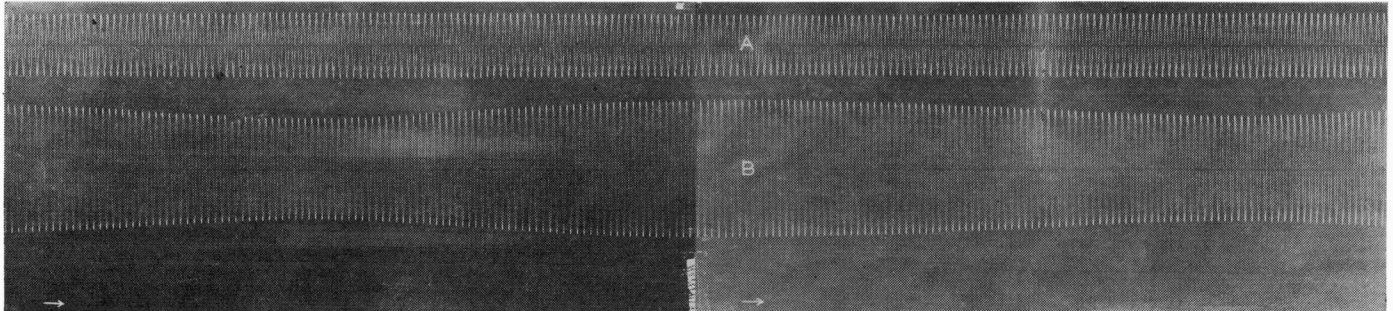


Figure 11. Determination of x_d and x_q by a "slip test"

Balanced voltages applied to armature, field open and rotated near synchronous speed

A—Armature voltage
B—Armature current

$$x_d = \frac{\text{maximum voltage}}{\text{minimum current}}$$

$$x_q = \frac{\text{minimum voltage}}{\text{maximum current}}$$

circuits are open or closed because, as defined, the steady-state, constant-magnitude, rotating fundamental of the armature magnetomotive force is always stationary with respect to the poles. Since there is no rate of change of magnetomotive force when viewed from the field structure, there will be no currents induced in any field winding, so that the armature magnetomotive force meets a constant permeance, the value of which is determined only by the air gap and magnetic circuit and not by the magnetomotive force of any field winding. Theoretically, any problem involving transformer circuits can be solved in terms of the self- and mutual inductances of windings. Likewise, any synchronous machine problem can theoretically be solved in terms of the synchronous reactances and the self- and mutual inductances of rotor circuits with themselves and with the armature. However, transformer-circuit calculations are greatly simplified and their accuracy improved by employing the short-circuit reactances between windings instead of self- and mutual inductances. A similar simplification can be made in synchronous machine calculations by the use of reactances which *take into account the transient effects of closed rotor windings*.

Thus, when conditions are changing (transient conditions) induced currents will flow in the closed rotor circuits, modifying the effective permeance of the armature flux paths and reducing the corresponding effective reactances. These reactances are analogous to the transformer short-circuit reactance discussed in part I which takes the effect of the closed secondary into account.

Consider the machine operating at normal speed with the unexcited main field winding closed, but with all other

just as the current in the transformer secondary of figure 3 resisted a sudden change in the primary magnetomotive force when a d-c voltage was suddenly impressed. Of course, the induced field current will die out eventually. However, during the first interval of time the field current will maintain the field linkage at practically zero value, and the armature flux is forced into the low permeance paths of the interpolar space, see figure 7, just as the transformer primary flux is forced into the leakage path between the primary and secondary in figure 3. The armature linkage per symmetrical, fundamental-frequency, armature ampere will be much less than under synchronous conditions and its value $\times \omega 10^{-8}$ gives direct transient reactance, x_d' , in ohms. This value divided by base ohms gives x_d' in per unit.

In order to visualize direct transient reactance from a test which might actually be conducted, it will again be necessary to conduct one which is different from the conditions used to define the reactance. Since we are now dealing with transient quantities, it will also be necessary to employ an oscillograph rather than instruments. Just as in the case of direct synchronous reactance, superposition allows a determination of direct transient reactance from a 3-phase short-circuit test; for in transient as well as steady short circuit the rotating fundamental component of armature magnetomotive force is directly in line with the poles tending to demagnetize them. Now it is possible to superpose the results of various *components* of the current in one circuit, considered separately, just as it is

* It is shown later that suddenly applied currents tend to contain d-c and harmonic components. Hence to obtain the current used in defining x_d' an unusual voltage would be required.

possible to superpose the results of *total currents* in separate circuits. For indeed, the linkage in any one circuit is directly proportional to the total current in each circuit, and hence, to the sum of any components into which each current may be divided in every circuit in the machine. Before short circuit, a certain steady field current existed which gave rise to a certain a-c terminal voltage. After, short circuit armature currents will suddenly start to flow. The sudden magnetomotive force appearing before the field will be resisted by a sudden increase in field current over its initial value. Also after short circuit the terminal voltage is zero, so the change of armature linkage must also be zero. Consequently, the armature linkage produced by the armature current plus that produced by the *increase* in field current has exactly the same value and is in opposite direction to the armature linkage produced by the initial value of field current acting by itself. Now this armature linkage, produced by the armature current plus the *increase* in field current acting together, is produced under exactly the same conditions which were used to define direct transient reactance; for indeed, the armature magnetomotive force appears suddenly in the direct axis and induces field current which opposes it, and the ratio of the symmetrical part† of the phase voltage to the symmetrical part of the phase current is x_d' . However,

nomena has been barely mentioned, the presence of d-c and harmonic components in the armature current. At the instant of short circuit each of the 3 armature phase circuits will be linked by some fraction of the flux produced by the main field excitation. Thus if the short circuit occurred at the instant the direct axis was in line with phase *a*, phase *a* would have maximum positive linkage and the linkages in phases *b* and *c* would be negative and half the maximum.

By the law of constant flux linkage, currents will flow in the armature and field to maintain the armature and field linkages at essentially constant value for a short time after short circuit. The jump in d-c field current to maintain the field linkage constant in the face of the magnetomotive force produced by the a-c, short-circuit armature current has been emphasized. However, the a-c armature current and d-c field current could produce constant armature phase linkages of only zero value. Hence, the armature current will contain d-c components to maintain the initial linkage. This results in a displaced wave of phase current such as shown in figure 13. An oscillogram of symmetrical short-circuit phase current when the phase linkage happened to be zero at the instant of short is shown in figure 12. Here no d-c component was necessary to maintain zero linkage.

Figure 12. Nearly symmetrical short-circuit armature current on a 35,000-kva 25-cycle water-wheel generator

A— Armature voltage
B— Armature current

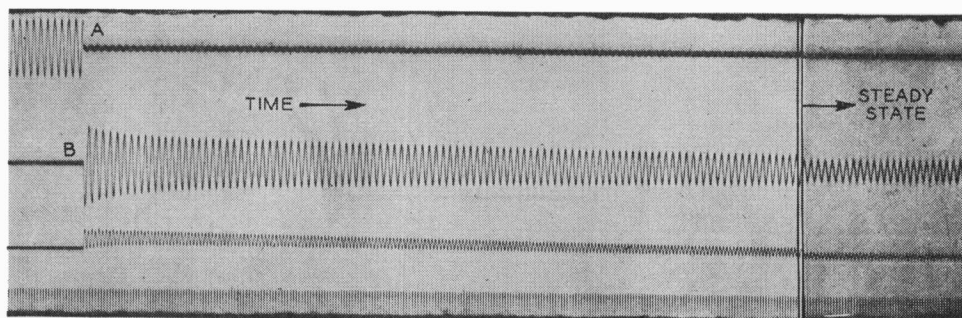
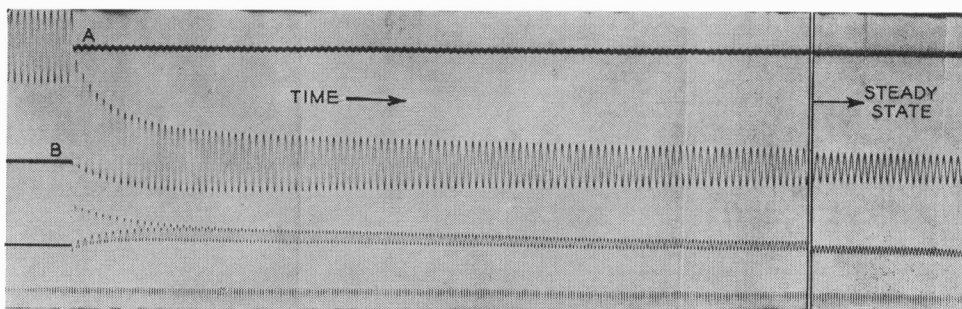


Figure 13. Nearly pure asymmetrical short-circuit armature current on the same machine as figure 12

A— Armature voltage
B— Armature current



this voltage has exactly the same magnitude as the open-circuit voltage which existed before short circuit. Therefore, the ratio of the voltage before short circuit to the symmetrical part of the armature current immediately after short circuit is direct transient reactance, x_d' , in ohms.

In the foregoing analysis one important part of the phe-

† The symmetrical part of a wave is the part symmetrical about the axis of zero deflection, that is the residue after the d-c component has been subtracted.

The d-c component of armature current results in a stationary magnetomotive force in the air gap. This would produce fundamental-frequency linkage in the field did not a fundamental-frequency field current flow to prevent it. The fundamental-frequency field current gives a pulsating magnetomotive force stationary with respect to the poles which may be resolved into 2 fundamental, magnetomotive-force waves rotating, one forward and one backward with respect to the poles at synchronous

speed. The backward rotating component is stationary with respect to the armature and opposes the d-c armature magnetomotive force. The forward rotating component is traveling at twice normal speed with respect to the armature so that a second-harmonic component of armature current must flow to prevent a change of armature linkage. This current will also have its effect on the field and so on. The purpose of this discussion is to complete, in a measure,

current will be induced in the amortisseur winding resisting the change in flux in addition to the current induced in the main field winding. Since the amortisseur winding is closer to the air gap than the main field, the permeance of the air-gap leakage-flux paths under this condition is still less than the corresponding permeance in the case of direct transient reactance. The corresponding armature linkage and reactance are also less. Invariably, the per-unit re-

Table I. Components of Synchronous Machine Reactances

Symbol	Name of Reactance	Position of Magnetomotive-Force Fundamental	Magnitude of Magnetomotive-Force Fundamental	Synchronous Motors				Turboalternator	
				General Purpose	Heavy Starting Duty	Synchronous Condensers	Water-Wheel Generator**	Solid Rotor	Laminated Rotor
x_d	Direct synchronous	Directly over pole	Constant	{ 0.80 0.65 } 1.60 avg	{ 0.60 } 1.15 avg 1.15 avg 1.15 avg
x_q	Quadrature synchronous	Midway between poles	Constant	{ 1.5 0.90 } 1.00 avg	{ 1.25 } 1.0 avg 1.0 avg 1.0 avg
x_d'	Direct transient	Directly over poles	Changing	{ 0.60 0.50 } 1.00 avg	{ 0.40 } 0.25 0.25 0.25
x_q'	Quadrature transient	Midway between poles	Changing	{ 1.10 0.70 } 1.00 avg	{ 0.80 } 0.13 0.13 0.13
x_d''	Direct subtransient	Directly over pole	Changing	{ 0.25 0.15 } 0.25	{ 0.20 } 0.08 0.08 0.08
x_q''	Quadrature subtransient	Midway between poles	Changing	{ 0.45 0.35 } 0.25	{ 0.35 } 0.15 0.15 0.15
x_2	Negative sequence	Moving relative to poles	Changing in each axis	{ 0.30 0.12 } 0.25	{ 0.25 } 0.10 0.10 0.10
x_0	Zero sequence	No fundamental		{ 0.50 0.30 } 0.35	{ 0.45 } 0.20 0.20 0.20
				{ 0.25 0.11 } 0.25	{ 0.20 } 0.08 0.08 0.11
				{ 0.45 0.27 } 0.32	{ 0.40 } 0.13 0.13 0.21
				{ 0.04 0.02 } 0.04	{ 0.02 } 0.01 0.01 0.01
				{ 0.27 0.15 } 0.10	{ 0.21 } 0.08 0.08 0.08

* x_0 varies from about 15 per cent to 60 per cent of x_d'' , depending upon winding pitch.

** These values apply to machines with amortisseur windings. For water-wheel generators without amortisseur windings, $x_d'' = x_d'$, $x_q'' = x_q' = x_q$. x_2 will be an average of x_d'' and x_q'' and x_0 will be unchanged.

the physical picture of the phenomena on short circuit. It is evident that the different components of current, existing for different purposes, meet different permeances. Consequently, direct axis transient reactance is defined for only the symmetrical, fundamental-frequency part of the armature current.

The actual test to determine the initial value of short-circuit current could be conducted by taking enough oscillograms of the current, till one was obtained which was symmetrical, or by graphically determining the symmetrical part of an asymmetrical wave, then drawing a line through the peaks of the successive decaying armature current waves and project it back to the time of short circuit neglecting the first few peaks of current which will be discussed under subtransient reactance. The initial value of short-circuit armature current may then be estimated as in the dotted line of figure 14 which was obtained from the oscillogram figure 12. Of course, if the peak value of the current is used, the peak value of the voltage must also be used.

Direct Subtransient Reactance x_d''

In defining direct transient reactance all field windings except the main field winding in the direct axis were assumed to be open. An amortisseur winding as shown in figure 9 may be present in both the direct and quadrature axes. If it is, and armature voltages are suddenly applied to the machine so that armature currents increase rapidly,

sistance of the amortisseur winding is considerably larger than the per-unit resistance of the main field, so that the induced amortisseur currents tend to die out much more rapidly. The high resistance is used to get the high induction-motor torque required for starting or to produce effective damping if the purpose of the amortisseur is to damp out oscillations. This rapid decay gives rise to the term subtransient reactance, indicating a very high speed phenomenon while transient reactance is the lower speed phenomenon caused by the main field windings.

Let the field structure be rotated mechanically at synchronous speed with the main field winding unexcited but closed, and all other field windings closed. If terminal voltages are now impressed at such an instant that positive-phase-sequence currents are produced, an armature magnetomotive force will be set up which rotates synchronously with its axis directly in line with that of the poles and amortisseur winding currents will be induced resisting the establishment of armature flux. The armature flux is then forced into the very low permeance air-gap leakage path as shown in figure 9 and the net armature linkage per armature ampere under this condition $\times \omega 10^{-8}$ gives direct subtransient reactance in ohms. This value divided by base ohms is x_d'' in per unit.

The test procedure described for determining direct transient reactance may also be used to determine direct subtransient reactance. If a closed amortisseur winding is present, the short-circuit current will have a form such as shown in figure 12 or 13. The same argument used in

deriving the expression for x_d' from the short-circuit test leads to the obvious result that x_d'' is the ratio of the open-circuit voltage existing before short circuit to the symmetrical part of the armature current existing immediately after short circuit. This value of the current may then be obtained by projecting the line connecting the peaks of the short-circuit current back to the zero axis as shown in figure 14. It has been pointed out that the amortisseur-winding currents die out very rapidly, so that their effect soon becomes negligible. Hence, x_d' may be determined from the same test by projecting the line connecting the more slowly decaying peaks of current back to the zero axis as shown.

Quadrature Transient Reactance x_q'

Consider the machine to be operated as in the case of direct transient reactance except that the terminal voltages are applied at a different instant so that the axis of the suddenly appearing armature magnetomotive force is now directly in line with the quadrature axis. Now if there are no closed rotor circuits in addition to the main field winding, the armature flux will pass through the main field winding from one side to the other as shown in figure 8. The net field linkage due to this flux is zero and hence no main-field-winding currents will be induced tending to distort the flux. The armature linkage under this condition per armature ampere $\times \omega 10^{-8}$ gives quadrature transient reactance in ohms. This value divided by base ohms gives x_q' in per unit.

Since the permeance presented to the armature magnetomotive force under this condition is exactly the same as

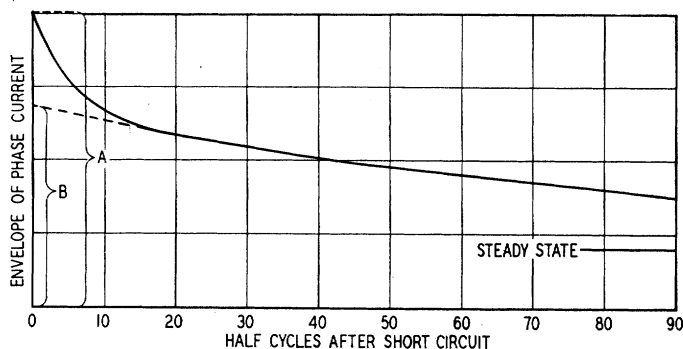


Figure 14. Determination of x_d' and x_d'' from test oscillogram, figure 12

Curve shows envelope of symmetrical short-circuit phase-current wave. x_d'' inversely proportional to initial current A. Dotted curve shows envelope of symmetrical current after high-speed transient has been subtracted. x_d' inversely proportional to B

that presented under the conditions defining quadrature synchronous reactance, the armature linkage per ampere will be the same in both cases, and the quadrature transient reactance is exactly equal to the quadrature synchronous reactance. This is literally true if the field structure is perfectly laminated. If the machine has a solid

rotor, currents will be induced in the rotor iron resisting the change in flux so that x_q' will be less than x_q .

Quadrature Subtransient Reactance x_q''

If an amortisseur winding exists in the quadrature axis, and terminal voltages are suddenly applied such that fundamental-frequency positive-sequence currents flow so that the resultant rotating armature magnetomotive force is in line with the quadrature axis of the mechanically rotated field structure, current will be induced in the amortisseur winding tending to maintain its linkage zero and forcing the armature flux into the very low permeance air-gap path as shown in figure 10. Under this condition the armature linkage per armature ampere $\times \omega 10^{-8}$ gives quadrature subtransient reactance in ohms. This value divided by base ohms gives x_q'' in per unit.

No simple convenient method of test exists to aid in visualizing this reactance. Methods of test exist^{10,11} but the mathematical analysis, required for establishing them, complicates the physical picture.

Negative-Sequence Reactance x_2

If terminal voltages are impressed such that fundamental-frequency negative-sequence currents flow while the unexcited field structure with all field windings closed is rotated mechanically at synchronous speed in the forward direction, the armature magnetomotive force will be traveling in the opposite direction at synchronous speed and hence will be traveling with twice synchronous speed with respect to the field structure. Then alternating currents of twice normal frequency will be induced in the rotor circuits, holding the field linkage at essentially zero value. This case is analogous to the a-c excited, short-circuited transformer of figure 3 for in the transformer an a-c magnetomotive force attempts to force flux through the short-circuited secondary and alternating currents are induced in the secondary holding the secondary linkage zero. In the case at hand the armature flux is forced into the very low permeance air-gap paths. Of course the actual permeance of the air-gap path varies as a function of time, for the armature magnetomotive force meets alternately the direct and quadrature axis. Under this condition the average armature linkage per armature ampere $\times \omega 10^{-8}$ gives the negative-sequence reactance in ohms. This value divided by base ohms is x_2 in per unit.

The permeance presented to the armature magnetomotive force at the instant that the magnetomotive force is over the axis of the poles is evidently essentially the same as the permeance presented under direct subtransient conditions and the effective reactance if the permeance were constant at this value would be x_d'' . However, an instant later the magnetomotive force is directly over the quadrature axis and at this instant the effective reactance is x_q'' . Hence, the average value of x_2 must lie between x_d'' and x_q'' . For most purposes the value of x_2 is taken as the arithmetic average of x_d'' and x_q'' .

$$x_2 = \frac{x_d'' + x_q''}{2}$$

The conditions used to define negative-sequence reactance may also be used to measure it. It is simply necessary to rotate the rotor mechanically at normal speed with the main field winding unexcited but short circuited, and to apply terminal voltages such that fundamental-frequency negative-sequence currents flow in the armature. The ratio of the average phase voltage to the average phase current under this condition gives x_2 in ohms. Tests may also be made with the rotor stationary²¹ but here again, they are of little use for visualization.

Zero-Sequence Reactance x_0

Zero-sequence currents are defined as currents which would flow if the 3 line terminals of a Y-connected machine were joined and single-phase voltage applied between the neutral and the terminals. Under this condition currents will flow in the 3 armature windings and will be of the same time phase. If such currents be impressed on a synchronous machine, no space fundamental of air-gap magnetomotive force will be produced so that the motion of the rotor is unimportant and the reactance is very small. The 3-phase armature windings will, in general, set up a third space harmonic of air-gap magnetomotive force which is stationary but pulsating. Currents will be induced in closed rotor circuits opposing this harmonic magnetomotive force and forcing the armature flux due to it into the low-permeance air-gap paths so that this reactance is indeed very small. The actual value depends greatly on winding pitch, being a minimum at $2/3$ pitch in a 3-phase

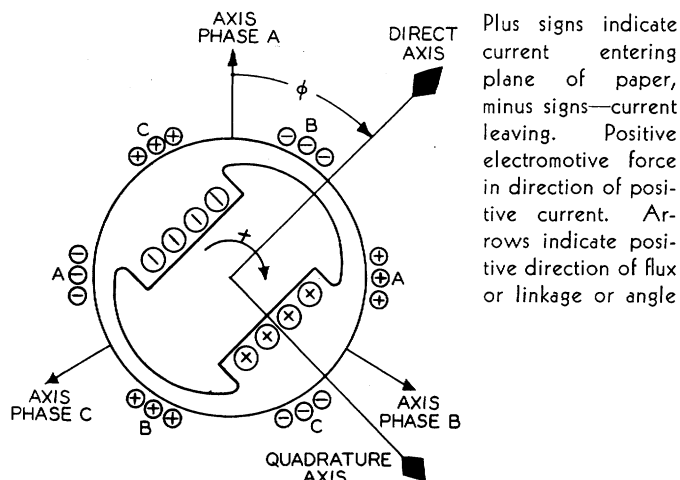


Figure 15. Conventions for positive angles, currents, electromotive forces, linkages, and rotation

winding. For at this pitch each slot contains coil sides of adjacent phases carrying currents in time phase but opposite direction. Consequently the net air-gap magnetomotive force of the slot is zero and the only flux produced in addition to end winding leakage is the slot leakage flux between the 2 conductors with opposite currents.

If zero-sequence currents be applied, the ratio of the phase linkage to phase current $\times \omega 10^{-8}$ gives zero-sequence reactance in ohms. This value divided by base

ohms is x_0 in per unit.

Zero-sequence currents do not produce the same slot, end winding, or differential air-gap leakage fluxes as positive-sequence currents so that x_1 is not included in x_0 .

The conditions used to define zero-sequence reactance are easily reproduced in an actual test, and if the machine were operated as described, the ratio of the voltage applied between the neutral and the 3 connected terminals to the current in one phase gives zero-sequence reactance in ohms.

Summary

The various synchronous machine armature reactances have been discussed from a number of different viewpoints, each designed to enhance the physical picture: (1) linkage per ampere, (2) effective permeance, (3) a-c voltage per a-c ampere, (4) test and (5) figures illustrating the flux paths. The foregoing discussion is summarized in table I which gives the various pertinent factors influencing each reactance, and typical numerical values.

Part III. Analytical Approach to Synchronous Machine Reactances

In the preceding parts a physical picture of certain arbitrary reactances has been built up. The uninitiated reader is, no doubt, in a quandary as to why these particular arbitrary definitions were selected. Furthermore, he is probably at a loss as to how to use such reactances in actual calculations of performance. It is not the purpose of this paper to present the whole of synchronous-machine analysis. However, the concepts of synchronous-machine reactance would be most incomplete without some idea as to their purpose and use. The following derivation is made from a most fundamental starting point and it will serve as a sound basis for proceeding with synchronous machine analysis as well as for broadening the physical picture already developed. Values of the various reactances are obtained in terms of the self- and mutual inductances of the various windings.

Fundamental Synchronous Machine Parameters

It is proposed to set up the differential equations of the synchronous machine in terms of the fundamental parameters of the machine, the self- and mutual inductances of the various circuits.

The self-inductance of an armature phase depends upon the position of the rotor, being a maximum when the direct axis is lined up with the magnetic axis of the phase and a minimum when the quadrature axis is in line with it. The variation of the self-inductance between these 2 positions can be calculated from design data or it may be determined from tests. It is found by both means that the variation in practical machines is approximately sinusoidal as shown in figure 16. Since the maximum value will obtain whether the pole which is before the phase is normally north or normally south in magnetic polarity, the self-

inductance will vary through 2 maximum values if the rotor is rotated through 360 electrical degrees. Since it will have some positive value at every position of the poles, it will be of the form,

$$L_a = L_0 + M_0 \cos 2\phi \quad L_0 > M_0 \quad (22)$$

where L_0 is the average and M_0 the amplitude of variation of the phase self-inductance in henries, (M_0 is not a mutual inductance as used here). ϕ is the electrical angle between the magnetic axis of the phase and the direct axis as shown in figure 15. This variation can be visualized by applying the concepts of part I to figure 15 and noting how the linkage per ampere of a phase varies as the rotor moves.

If ϕ be measured from the axis of phase a then 22 gives the self-inductance of phase a , for at $\phi = 0$, L_a is a maximum. Obviously, in a symmetrical machine, the self-

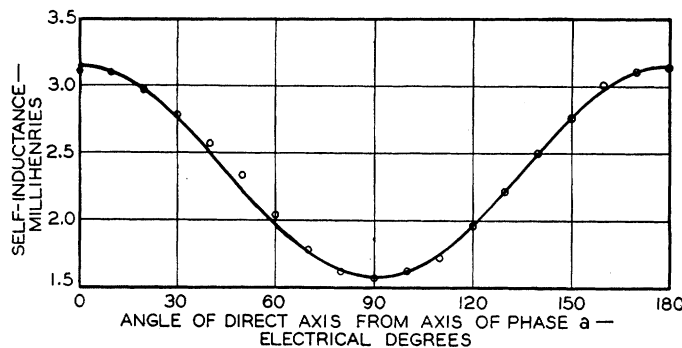


Figure 16. Variation of self-inductance of armature phase a with position of the field

The test points, taken on a 4-pole 15-kva 220-volt 1,800-rpm synchronous motor, compare favorably with the curve showing the theoretical variation of an ideal machine,

$$L_a = L_0 + M_0 \cos 2\phi = 2.36 + 0.796 \cos 2\phi \text{ (millihenries)}$$

inductance of each phase will have the same form except the maximum value will occur at a different value of ϕ . That of phase b will be a maximum when $\phi = 120$ electrical degrees,

$$L_b = L_0 + M_0 \cos 2(\phi - 120) = L_0 + M_0 \cos (2\phi + 120) \quad (23)$$

Likewise, the self-inductance of phase c will be a maximum when $\phi = -120$ degrees,

$$L_c = L_0 + M_0 \cos 2(\phi + 120) = L_0 + M_0 \cos (2\phi - 120) \quad (24)$$

The mutual inductances between armature phases will also be functions of the rotor position. Positive currents are logically defined in the same relative direction in each of the phases as shown in figure 15. Thus, positive current in phase b tends to produce linkage in phase a in the opposite direction to that produced by positive current in phase a (see figure 15). Hence, the sign of the mutual inductance between phases a and b is negative regardless of rotor position. The absolute magnitude of this mutual inductance will have a minimum value when the axis of the poles is midway between the phases (for phases a and b , at $\phi = 60$ degrees), and a maximum value when the rotor has advanced 90 degrees from the position of minimum

value. Hence, the mutual inductance between 2 phases also varies through 2 maximum values for a 360 degree rotation of the rotor. Experiments and calculations both show that the mutual inductance also varies approximately sinusoidally with ϕ as shown by the test on a particular machine in figure 17. Furthermore, experiments show that the amplitude of variation of mutual inductance in

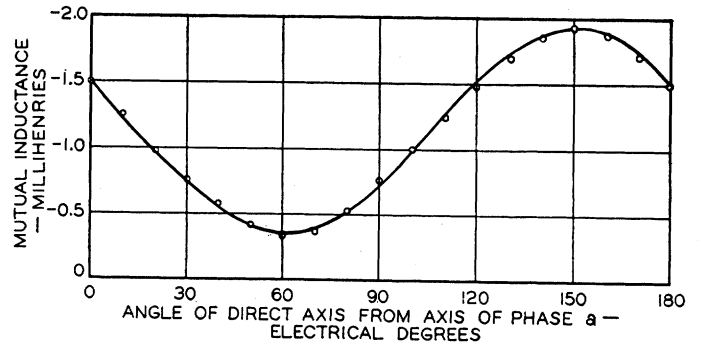


Figure 17. Variation of mutual inductance between armature phases a and b with position of the field

The test points taken on the same machine as in figure 16 are close to the curve which shows theoretical variation for an ideal machine

$$\begin{aligned} M_{ab} &= -M_s + M_0 \cos (2\phi - 120^\circ) \\ &= -1.14 + 0.796 \cos (2\phi - 120^\circ) \text{ millihenries} \end{aligned}$$

practical machines is nearly the same as the amplitude of variation of self-inductance (see figures 16 and 17). Then, if M_s is the average value of the mutual inductance between phases, the same M_0 used in equations 22, 23, and 24 is the amplitude of pulsation of the mutual inductance. Hence, the mutual inductance between phases a and b is

$$\begin{aligned} M_{ab} &= -[M_s - M_0 \cos 2(\phi - 60)] \\ &= -M_s + M_0 \cos (2\phi - 120) \quad M_s > M_0 \quad (25) \end{aligned}$$

Likewise, the minimum absolute magnitude of the mutual inductance between phases b and c will occur at $\phi = 0$ and

$$M_{bc} = -[M_s - M_0 \cos 2\phi] = -M_s + M_0 \cos 2\phi \quad (26)$$

Also

$$M_{ac} = -[M_s - M_0 \cos 2(\phi + 60)] = -M_s + M_0 \cos (2\phi + 120) \quad (27)$$

A complete analysis would, of course, include the effect of an arbitrary number of field circuits. However, the complexity of the equations which arise obscures the physical picture so that in this analysis no other rotor circuits except the main, direct-axis, field winding are assumed. Inspection of figure 15 shows that M_a , the maximum mutual inductance between phase a and the main field winding will exist when the direct axis is in line with the magnetic axis of phase a , at $\phi = 0$, and its value will be equal but negative at $\phi = 180$ degrees. Test and calculation show that it varies approximately sinusoidally between these limits, so

$$M_{ad} = M_d \cos \phi \quad (28)$$

The mutual inductance between phase b and the main field winding will have the same form but will be a maximum at $\phi = 120$ degrees, so

$$M_{ba} = M_d \cos(\phi - 120) \quad (29)$$

Likewise,

$$M_{ca} = M_d \cos(\phi + 120) \quad (30)$$

Fundamental Synchronous Machine Equations

The assumptions which have been made are:

1. Saturation, hysteresis, and eddy currents are negligible.
2. Self- and mutual inductances vary sinusoidally as the rotor moves.

On the basis of the fundamental, experimental laws of Faraday, equation 1, and Ohm, the voltage, e , impressed on an armature phase may be expressed in terms of the phase resistance, r , phase linkage, ψ and phase current i , in phase a as

$$e_a = r i_a + \frac{d}{dt} \psi_a \times 10^{-8}$$

noting the time derivative operator, d/dt , by p ,

$$e_a = r i_a + p \psi_a \times 10^{-8} \quad (31)$$

and for phases b and c ,

$$e_b = r i_b + p \psi_b \times 10^{-8} \quad (32)$$

$$e_c = r i_c + p \psi_c \times 10^{-8} \quad (33)$$

A similar equation may be written for the main field winding in terms of its applied voltage E_d , current I_d , linkage Ψ_d and resistance R_d .

$$E_d = R_d I_d + p \Psi_d \times 10^{-8} \quad (34)$$

Before these differential equations (31 to 34) can be solved for the currents in terms of the applied voltages it is necessary, of course, to express the linkages as functions of the currents in all circuits. Thus, from the definitions of self- and mutual inductance, equations 3 and 4, the linkage in phase a is

$$\psi_a = \{L_a i_a + M_{ab} i_b + M_{ac} i_c + M_{ad} I_d\} 10^8$$

From equations 22, 25, 27, and 28,

$$\psi_a = \{i_a [L_0 + M_0 \cos 2\phi] + i_b [-M_s + M_0 \cos(2\phi - 120)] + i_c [-M_s + M_0 \cos(2\phi + 120)] + I_d M_d \cos \phi\} 10^8$$

$$\psi_a = \{L_0 i_a - M_s [i_b + i_c] + M_0 [i_a \cos 2\phi + i_b \cos(2\phi - 120) + i_c \cos(2\phi + 120)] + I_d M_d \cos \phi\} 10^8 \quad (35)$$

Likewise, from equations 23, 25, 26, and 29,

$$\psi_b = \{L_0 i_b - M_s [i_a + i_c] + M_0 [i_c \cos 2\phi + i_a \cos(2\phi - 120) + i_b \cos(2\phi + 120)] + I_d M_d \cos(\phi - 120)\} 10^8 \quad (36)$$

and from equations 24, 26, 27, and 30,

$$\psi_c = \{L_0 i_c - M_s [i_a + i_b] + M_0 [i_b \cos 2\phi + i_c \cos(2\phi - 120) + i_a \cos(2\phi + 120)] + I_d M_d \cos(\phi + 120)\} 10^8 \quad (37)$$

If the main field winding self-inductance is L_{fd} —ob-

viously it is a constant—the main field linkage may be expressed in terms of all currents by considering figure 15 and equations 28, 29, and 30

$$\begin{aligned} \Psi_d &= M_{ad} i_a + M_{bd} i_b + M_{cd} i_c + L_{fd} I_d \\ &= M_d [i_a \cos \phi + i_b \cos(\phi - 120) + i_c \cos(\phi + 120)] + L_{fd} I_d \end{aligned} \quad (37a)$$

If equations 35, 36, 37, and 37a are substituted in equations 31 to 34, the fundamental simultaneous differential equations of the synchronous machine will have been set up from Faraday's experimental law, equation 1, and measurable self- and mutual inductances.

Theoretically, the resulting differential equations can be solved simultaneously for the currents with any given applied voltages and motion of the rotor either in transient or steady state. However, if the rotor is moving, ϕ is a function of time. Consequently, when the differentiations indicated in equations 31 to 34 are performed, terms such as the following arise,

$$\begin{aligned} \frac{d}{dt} \left[M_0 i_b \cos(2\phi - 120) \right] = \\ -2M_0 i_b \sin(2\phi - 120) \frac{d\phi}{dt} + M_0 \cos(2\phi - 120) \frac{di_b}{dt} \end{aligned}$$

When the resulting differential equations are combined simultaneously to get a differential equation in terms of one of the currents only, a multitude of terms, even more complex than the one shown, arise which include functions of time—such as $\cos 2\phi$ —as coefficients of the derivatives. The solution of these differential equations with variable coefficients as they stand is so difficult that it has never been carried out in a general case, and this is the reason for resorting to direct and quadrature axes, namely, in order to simplify the mathematics.

All are familiar with linear and nonlinear changes of variables in the calculus in order to simplify integration. Also, the change of co-ordinate axes such as from Cartesian to polar, or 3-dimensional Cartesian to spherical, etc., are nonlinear changes of variable, and rotation or translation of Cartesian reference axes are linear changes of variable which are commonplace. In the latter cases it is important to note that the number of degrees of freedom is not reduced and the new variables are independent of each other and related to the old ones by linear equations. Now if someone suggested that new, independent currents, flux linkages, etc., might be assumed, related to the old currents, flux linkages, etc., by linear equations which would facilitate the solution; we might throw up our hands in horror at this unorthodox procedure if we did not realize that the differential equations to be solved were really mathematical statements which can be manipulated at will so long as the fundamental rules of calculus, algebra, etc., are not violated. Consequently, it is sensible to take the view that the proposed transformation is a mathematical one and as long as it is eventually possible to return and find the old currents, linkages, etc., it is possible to proceed with confidence.

* This method of analysis is similar to that used by Park¹⁸ except for different sign conventions, but is not the same as that used by Park⁵ as it avoids the use of permeance coefficients. The sign conventions used here are those widely accepted in classical literature and were defined in part I.

Assume then, that someone has suggested the following new variables:

i_d, ψ_d, e_d —direct axis current, linkage, voltage

i_q, ψ_q, e_q —quadrature axis current, linkage, voltage

i_0, ψ_0, e_0 —zero axis current, linkage, voltage

which are linearly related to the old variables by the equations,

$$\left. \begin{aligned} i_d &= \frac{2}{3} \left\{ i_a \cos \phi + i_b \cos (\phi - 120) + i_c \cos (\phi + 120) \right\} \\ i_q &= -\frac{2}{3} \left\{ i_a \sin \phi + i_b \sin (\phi - 120) + i_c \sin (\phi + 120) \right\} \\ i_0 &= \frac{1}{3} \left\{ i_a + i_b + i_c \right\} \end{aligned} \right\} (38)$$

and

$$\left. \begin{aligned} \psi_d &= \frac{2}{3} \left\{ \psi_a \cos \phi + \psi_b \cos (\phi - 120) + \psi_c \cos (\phi + 120) \right\} \\ \psi_q &= -\frac{2}{3} \left\{ \psi_a \sin \phi + \psi_b \sin (\phi - 120) + \psi_c \sin (\phi + 120) \right\} \\ \psi_0 &= \frac{1}{3} \left\{ \psi_a + \psi_b + \psi_c \right\} \end{aligned} \right\} (39)$$

and

$$\left. \begin{aligned} e_d &= \frac{2}{3} \left\{ e_a \cos \phi + e_b \cos (\phi - 120) + e_c \cos (\phi + 120) \right\} \\ e_q &= -\frac{2}{3} \left\{ e_a \sin \phi + e_b \sin (\phi - 120) + e_c \sin (\phi + 120) \right\} \\ e_0 &= \frac{1}{3} \left\{ e_a + e_b + e_c \right\} \end{aligned} \right\} (40)$$

Actually, these new variables were conceived and developed by engineers^{1,2,3,4,5} by virtue of their very thorough and complete "physical picture" of the phenomena. For the present purpose, the physical picture and the physical reasoning back of the selection of such a set of transformations will be discussed after the results are obtained.

If the proposed change of variables enables solution of the equations, the actual physical phase currents, linkages, voltages, etc., can then finally be obtained from the following expressions for the phase quantities in terms of the new variables. Thus, solving simultaneously equations 38:

$$\left. \begin{aligned} i_a &= i_d \cos \phi - i_q \sin \phi + i_0 \\ i_b &= i_d \cos (\phi - 120) - i_q \sin (\phi - 120) + i_0 \\ i_c &= i_d \cos (\phi + 120) - i_q \sin (\phi + 120) + i_0 \end{aligned} \right\} (41)$$

and equations 39:

$$\left. \begin{aligned} \psi_a &= \psi_d \cos \phi - \psi_q \sin \phi + \psi_0 \\ \psi_b &= \psi_d \cos (\phi - 120) - \psi_q \sin (\phi - 120) + \psi_0 \\ \psi_c &= \psi_d \cos (\phi + 120) - \psi_q \sin (\phi + 120) + \psi_0 \end{aligned} \right\} (42)$$

* A hint to the student in solving, say, equations 40 to obtain equations 43 is to multiply the first one of equations 40 by $\cos \phi$, the second by $-\sin \phi$ and add. The trigonometric coefficients of e_b and e_c can be shown to be zero, that of e_a unity which gives a ready solution for e_a in terms of $e_d, e_q,$ and e_0 . Similar operations yield e_b and e_c .

and equations 40:

$$\left. \begin{aligned} e_a &= e_d \cos \phi - e_q \sin \phi + e_0 \\ e_b &= e_d \cos (\phi - 120) - e_q \sin (\phi - 120) + e_0 \\ e_c &= e_d \cos (\phi + 120) - e_q \sin (\phi + 120) + e_0^* \end{aligned} \right\} (43)$$

Now to proceed with the change of variable. Substitute equations 38 in equations 35, 36, and 37. Combine the resulting expressions for $\psi_a, \psi_b,$ and ψ_c according to equations 39 to get $\psi_d, \psi_q,$ and ψ_0 wholly in terms of i_d, i_q, i_0 the self- and mutual inductances and trigonometric expressions. The algebra and trigonometry of these transformations are formidable but if the student has sufficient fortitude he will find that the many multiple products of trigonometric functions are all reducible to single simple constants and he will be rewarded by the following very simple equations:

$$\psi_d = \left\{ M_d I_d + \left(L_0 + M_s + \frac{3}{2} M_0 \right) i_d \right\} 10^8 \quad (44)$$

$$\psi_q = \left\{ \left(L_0 + M_s - \frac{3}{2} M_0 \right) i_q \right\} 10^8 \quad (45)$$

$$\psi_0 = \left\{ (L_0 - 2M_s) i_0 \right\} 10^8 \quad (46)$$

These expressions are relations between the new variables, linkages, and currents, in the direct, quadrature, and zero axes. Note that each equation contains only variables in the same axis. Mathematically this means that the choice of new variables has been particularly appropriate and that the equations will be relatively simple to solve in terms of the new variables.

Now each equation physically is a relation between currents and linkage and the factors of proportionality are constants. Consequently it requires only a little imagination to see that the bracketed constants of proportionality are really equivalent inductances in the new system of variables. Furthermore, it is natural to suspect that the constant of proportionality in equation 44 is the inductance corresponding to direct synchronous reactance x_d , that of equation 45 to x_q , and that of equation 46 to x_0 . That this is true can be proved quickly by applying the conditions defining these reactances in part II to equation 35, and finding the linkage in phase a , ψ_a , per unit current in phase a , i_a . Either of the other phases might be used.

Direct Synchronous Reactance x_d

The conditions specified are:

1. Positive-sequence steady-state currents applied to the armature

$$\left. \begin{aligned} i_a &= i \cos \omega t \\ i_b &= i \cos (\omega t - 120) \\ i_c &= i \cos (\omega t + 120) \end{aligned} \right\} (46a)$$

and

$$i_b + i_c = -i_a$$

2. Field circuit open,

$$I_d = 0$$

3. Field rotated at constant synchronous speed so that the direct axis is in line with the axis of the armature magnetomotive-force wave,

$$\phi = \omega t$$

Substitute these conditions in equation 35.

$$\psi_a = \{i \cos \omega t (L_0 + M_s) + M_0 i [\cos \omega t \cos 2\omega t + \cos(\omega t - 120) \times \cos(2\omega t - 120) + \cos(\omega t + 120) \cos(2\omega t + 120)]\} 10^8$$

which, upon trigonometric reduction becomes

$$\begin{aligned} \psi_a &= \left[\left(L_0 + M_s + \frac{3}{2} M_0 \right) i \cos \omega t \right] 10^8 \\ &= \left(L_0 + M_s + \frac{3}{2} M_0 \right) i_a \times 10^8 \end{aligned}$$

and as defined, x_d in ohms is the armature linkage per armature ampere times $\omega 10^{-8}$ so

$$x_d = \frac{\psi_a}{i_a} \omega 10^{-8} = \left(L_0 + M_s + \frac{3}{2} M_0 \right) \omega \quad (47)$$

Quadrature Synchronous Reactance x_q

The conditions specified are:

1 and 2 are the same as under x_d .

3. Field rotated at constant speed so that the quadrature axis is in line with the axis of the armature magnetomotive-force wave,

$$\phi = \omega t - 90^\circ$$

Then equation 35 becomes

$$\begin{aligned} \psi_a &= \{i \cos \omega t (L_0 + M_s) + M_0 i [\cos \omega t \cos 2(\omega t - 90) + \cos(\omega t - 120) \cos \{2(\omega t - 90) - 120\} + \cos(\omega t + 120) \cos \{2(\omega t - 90) + 120\}]\} 10^8 \\ &= \{i \cos \omega t (L_0 + M_s) - M_0 i [\cos \omega t \cos 2\omega t + \cos(\omega t - 120) \cos(2\omega t - 120) + \cos(\omega t + 120) \cos(2\omega t + 120)]\} 10^8 \end{aligned}$$

which, by comparison with the previous example is

$$\psi_a = \left(L_0 + M_s - \frac{3}{2} M_0 \right) i \cos \omega t \times 10^8$$

and

$$x_q = \left(L_0 + M_s - \frac{3}{2} M_0 \right) \omega \quad (48)$$

Zero-Sequence Reactance x_0

The conditions specified are:

1. Zero-sequence, steady-state currents applied to the armature,

$$i_a = i_b = i_c = i \cos \omega t$$

2. Effect of field circuits, negligible,

$$I_d = 0$$

3. Motion of the rotor immaterial as will be evident,

$$\phi = \phi$$

* The sign is negative because the usual convention is to designate the interpolar space ahead of the poles as the quadrature axis. This is explicitly required by the signs in equations 38, 39, and 40.

Equation 35 becomes,

$$\psi_a = \{i \cos \omega t [L_0 - 2M_s] + M_0 i \cos \omega t [\cos \phi + \cos(\phi - 120) + \cos(\phi + 120)]\} 10^8$$

The last term is obviously zero regardless of how ϕ varies, so

$$\psi_a = \{[L_0 - 2M_s] i \cos \omega t\} 10^8$$

and

$$x_0 = (L_0 - 2M_s) \omega \quad (49)$$

Equations 44, 45, and 46 then become,

$$\left. \begin{aligned} \psi_d &= \left[M_d I_d + \frac{x_d i_d}{\omega} \right] 10^8 \\ \psi_q &= \frac{x_q i_q}{\omega} \times 10^8 \\ \psi_0 &= \frac{x_0 i_0}{\omega} \times 10^8 \end{aligned} \right\} \quad (50)$$

Machine Equations in the Direct, Quadrature, and Zero Axes

Before deriving an expression for x_d' the machine equations in terms of the direct-, quadrature-, and zero-axis quantities will be determined. They will then be employed to obtain x_d' to illustrate the use of these equations rather than using the method just applied to get x_d , x_q , and x_0 . The following derivation was given by Park⁶ with different sign conventions. Combining the fundamental phase voltage equations 31, 32, and 33 with the definition of e_d , equation 40,

$$e_d = \frac{2}{3} \left\{ \cos \phi (r i_a + p \psi_a \times 10^{-8}) + \cos(\phi - 120) \times (r i_b + p \psi_b \times 10^{-8}) + \cos(\phi + 120) (r i_c + p \psi_c \times 10^{-8}) \right\}$$

and the definition of i_d , equation 38, allows a reduction to

$$e_d = r i_d + \frac{2}{3} \left\{ \cos \phi p \psi_a + \cos(\phi - 120) p \psi_b + \cos(\phi + 120) p \psi_c \right\} \times 10^{-8} \quad (51)$$

The bracketed term is unfamiliar; however, differentiate ψ_d , equation 39,

$$\begin{aligned} p \psi_d &= \frac{2}{3} \left\{ \cos \phi p \psi_a + \cos(\phi - 120) p \psi_b + \cos(\phi + 120) p \psi_c \right\} - \\ &\quad \frac{2}{3} \left\{ \psi_a \sin \phi + \psi_b \sin(\phi - 120) + \psi_c \sin(\phi + 120) \right\} p \phi \quad (52) \end{aligned}$$

The first bracketed term of this expression is identical with the bracketed term in equation 51. However, the last bracketed term in equation 52 is, by comparison with the

definition of ψ_q , equation 39, $\psi_q p\phi$. Therefore the bracketed expression in equation 51 is

$$p\psi_d - \psi_q p\phi$$

and

$$e_d = r i_d + (p\psi_d - \psi_q p\phi) 10^{-8}$$

By similar substitutions Park⁶ shows that

$$e_q = r i_q + (p\psi_q + \psi_d p\phi) 10^{-8} \quad (53)$$

and

$$e_0 = r i_0 + p\psi_0 \times 10^{-8}$$

Now, ψ_d , ψ_q , and ψ_0 are known in terms of currents and measurable reactances, equations 50. Consequently, equations 50, 53, and 34 may be used to solve for i_d , i_q , i_0 , and I_d if the applied voltages e_d , e_q , e_0 , and E_d are known, and the motion of the rotor is known. The phase currents may then be obtained from equations 41. This is one of the first steps in a complete method of synchronous machine analysis as given by Park.⁶

Direct Transient Reactance x_d'

The conditions used to define direct transient reactance were:

1. Positive-phase-sequence currents suddenly applied to the armature, expressions for which are given in equation 46a.
2. Field circuit closed, not excited. By the constant linkage law

$$\Psi_d = 0$$

3. Field rotated at constant speed so that the direct axis is in line with the peak of newly established armature magnetomotive-force wave.

$$\phi = \omega t$$

If conditions 1 and 3 be put in equation 38, the direct-, quadrature-, and zero-axis currents as functions of time are found to be

$$i_d = i$$

$$i_q = 0$$

$$i_0 = 0$$

Comparison of equation 50 with condition 1 shows that

$$\psi_d = \left[M_d I_d + \frac{x_d}{\omega} i_d \right] 10^8 \quad (54)$$

$$\psi_q = 0$$

$$\psi_0 = 0$$

Comparing equation 37a with i_d in equation 38 shows that

$$\Psi_d = L_{fd} I_d + \frac{3}{2} M_d i_d \quad (55)$$

which gives a ready solution for I_d on application of condition 2, thus

$$I_d = -\frac{3}{2} \frac{M_d}{L_{fd}} i_d$$

Then, equation 54 gives ψ_d in terms of i_d

$$\psi_d = \left[\frac{x_d}{\omega} - \frac{3}{2} \frac{M_d^2}{L_{fd}} \right] i_d 10^8 \quad (56)$$

Since $\psi_q = \psi_0 = i_q = i_0 = 0$, equations 41 and 42 show that

$$\frac{\psi_a}{i_a} = \frac{\psi_d}{i_d}$$

Therefore, the armature linkage per armature ampere under this condition is

$$\frac{\psi_a}{i_a} = \left[\frac{x_d}{\omega} - \frac{3}{2} \frac{M_d^2}{L_{fd}} \right] 10^8 = \left[L_0 + M_s + \frac{3}{2} M_0 - \frac{3}{2} \frac{M_d^2}{L_{fd}} \right] 10^8$$

and the direct transient reactance is

$$\begin{aligned} x_d' &= \frac{\psi_a}{i_a} \omega 10^{-8} = \left[L_0 + M_s + \frac{3}{2} M_0 - \frac{3}{2} \frac{M_d^2}{L_{fd}} \right] \omega \\ &= x_d - \frac{3}{2} \frac{M_d^2}{L_{fd}} \omega \end{aligned} \quad (57)$$

The similarity between the expression for x_d' and the short-circuit transformer leakage reactance equation 19 is striking.

$$x_{sc} = \omega L_1 - \frac{M'^2}{L_2'} \omega \quad (19)$$

The analogy is obvious. Recall, for instance that the physical argument used in part II in discussing x_d' which was based on short-circuit transformer phenomena. The presence of the $3/2$ may be physically explained on the basis that all 3 phase currents contribute to the armature magnetomotive force attempting to establish flux in the field, whereas in the transformer only one primary coil was excited, however, space does not permit going through this interesting physical argument. Nevertheless, it is strongly recommended that the student make thoroughgoing physical explanations of the presence of the various terms in the equations for x_d , x_q , x_0 , and x_d' . Such a task will both require and greatly enhance the physical picture of the synchronous machine.

Per-Unit Quantities

In previous literature, mathematical developments such as given here have been carried out in the per-unit system even in literal form. In more complete derivations such procedure has advantage in the resulting simplicity of equations. However, for one not thoroughly familiar with the per-unit system, the nonhomogeneous appearance of equations is confusing. Consequently, in this educational paper, the derivations have been presented in classical notation, it being inadvisable to present 2 new viewpoints simultaneously. All equations in part III can be converted to equations in a consistent per-unit system simply by omitting all ω 's, 10^8 's, and 10^{-8} 's.

The major value of the per-unit system lies in numerical work. If one has solved a problem for a typical machine in per unit, the results apply to any size or rating of ma-

chine with similar per-unit parameters. Furthermore, the numbers in a per-unit calculation are of convenient size and the engineer can check their approximate values from experience and physical reasoning, thus reducing the chance for error.

Physical Picture

The final phase of the physical picture, which this paper attempts to build up, may now be developed from equations 53. We seek a concept of the fictitious direct-, quadrature-, and zero-axis windings which give rise to the rather unusual equations 53. The equations are unusual because of the presence of the $\psi_a p\phi$ and $\psi_a p\phi$ terms, the like of which do not occur in equations for the electromotive force in a circuit fixed in the observer's initial system.

Let us as observers, reside upon the rotating field structure and attempt to visualize the phenomena experienced in the direct axis. The armature phase currents, assumed to be positive phase sequence for this visualization, will produce a rotating sinusoidal magnetomotive-force wave which may be resolved into 2 space sinusoidal components, one in line with the direct axis, the other with the quadrature axis. The absolute and relative magnitudes of the components may, of course, vary with time. Now visualize a winding opposite the direct axis and rotating with it which has the same resistance as an armature phase, r . If this winding carried a current i_a it would produce a magnetomotive force in the direct axis, and i_a might be such a function of time that its magnetomotive force was always equal to the direct axis component of the magnetomotive force produced by phase currents. For instance, if *steady-state*, positive-sequence armature currents flowed, i_a would be direct current, producing a constant magnetomotive force in the direct axis. The voltage equation of such a fictitious winding would be, by Faraday's and Ohm's laws,

$$e_a = ri_a + p\psi_a \times 10^{-8}$$

for ψ_a is the component of armature linkage opposite the direct axis. But, this physical picture is not complete because of the absence of $\psi_a p\phi$. The reason is that we have attempted to replace a symmetrical series of armature windings moving past the field pole and relative to the observer with a velocity $p\phi$ by one *ordinary* winding stationary with respect to the field poles. Now note that the *sides* of the fictitious direct-axis winding lie in the quadrature axis and in a flux density proportional to ψ_a . If these sides of the winding are imagined to have an instantaneous velocity $p\phi$ relative to the poles and the observer, then they would have a motional electromotive force induced due to motion relative to the observer proportional to $\psi_a p\phi$, and, if the fictitious winding is chosen properly and ψ_a is positive ahead of the poles, equal to $-\psi_a p\phi$. The resulting equation would then be

$$e_a = ri_a + (p\psi_a - \psi_a p\phi)10^{-8}$$

The complete physical picture of the fictitious direct-axis winding is, then, a winding, similar to an armature phase, moving with the speed of the poles so that its magnetic

axis is always in line with the direct axis, but *extraordinary* to the extent that its coil sides have an instantaneous velocity relative to the field poles of $p\phi$. Study will show that this is equivalent to a commutated winding like, for instance, the armature winding of a d-c motor when viewed from the field poles. Furthermore, the motional voltage cannot be attributed to a velocity through flux for the flux is produced by the combined action of the moving field poles and stationary armature conductors.

Obviously the physical picture of the fictitious quadrature-axis winding is identical with that for the direct axis, except for location. The term $\psi_a p\phi$ appears with a positive sign since positive direct-axis linkage is "behind" the quadrature axis.

The physical picture of a zero-axis winding is unimportant. It may be a 3-phase winding, stationary on the armature, connected as described in part II under "Zero-Sequence Reactance."

Summary and Recommendation to Students

It is important to recall that the references, mentioned in the introduction and throughout the paper, furnish the basis for this paper. Part I is a presentation of the fundamental concepts of inductance such as has been developed and clarified in the advanced course in engineering¹⁴ of the General Electric Company by many lecturers. Part II includes the viewpoints presented by Shildneck,¹² some of the tests given by Park and Robertson,⁹ and some ideas not heretofore published. Part III incorporates methods similar to those used by Park,¹³ but with considerable emphasis on the physical picture.

This paper has been written specifically to bridge the gap existing between collegiate study and current synchronous-machine literature. Hence, it is appropriate to suggest a program to the student for further study leading to the actual analysis. The program suggested is essentially that which has been used in the *B*-class, electrical engineering section of the advanced course in engineering in training graduates, one year out of college. A knowledge of differential equations such as given by Doherty and Keller¹⁵ is necessary, and a knowledge of operational calculus¹⁵ is desirable before proceeding far in the program.

1. Reference 1 gives the theoretical proof of validity of direct- and quadrature-axis representation and a study of it gives an excellent physical picture of the actual fluxes, linkages, currents, etc., in the machine.

2. The student may then proceed directly to references 5 and 6 and study and use, the complete, powerful, analytical methods presented by Park therein. These references are extremely concise and considerable thinking will be required of the student to follow the analysis and extend it to new problems. However, the more thinking required the better will be his ultimate understanding. Either sign conventions may be employed. Those used in this paper probably agree with the students' previous training.

3. Concurrently with the study just recommended references 7, 8, and 9 should be used to extend the students'

knowledge of reactances and references 2, 3, and 4 should be studied when working any of the specific types of problems therein, to aid in forming a very clear, complete physical picture of the phenomena; a very important phase.

4. Having mastered the preceding ideas and methods, the student should be able to solve a large number of synchronous machine problems, granting sufficient engineering and mathematical ingenuity and available time, on the assumption of no saturation, hysteresis, or eddy currents. Then he may proceed with the more recent literature on the extensions of the theory and its associated reactances.

It should not be inferred from this paper that synchronous-machine problems are all of standard type, and solvable by standard methods. Quite to the contrary, new problems are arising all of the time which require analysis directly from fundamentals. For example, a few such problems which have become important recently are:

1. The effects of saturation on both steady-state and transient performance. In some cases the effect of saturation is even more pronounced than the effect of saliency.
2. The effects of eddy currents in the solid cylindrical rotor of turbine generators during transient conditions, mentioned in part II.
3. Analysis of phenomena contingent to internal faults in the machine. This problem involves analysis with nonsymmetrical windings; additional complications arise due to dissymmetry, not only between phases but between the various pairs of poles on the armature and field.
4. Determination of reactances and performance of large double-winding generators.
5. Determination of performance with external capacitance. This problem requires an extension of the 2-reaction theory such as given by Crary.²⁰

As such novel problems are frequently of great consequence and a recourse to fundamentals is necessary for their solution, the paramount importance of a thorough knowledge of the fundamentals and their associated physical picture cannot be overemphasized.

Nomenclature

Subscripts

$a, b,$ and c refer to armature phases $a, b,$ and c of a machine.
 $d, q,$ and 0 refer to direct-, quadrature-, and zero-axis quantities.

Notation

i = current, amperes
 I_d = direct-axis main-field-winding current, amperes
 e = electromotive force, volts
 E_d = direct-axis main-field-winding electromotive force, volts
 ψ = flux linkage, lines times turns
 Ψ_d = direct-axis main-field-winding linkage, lines times turns
 L = self-inductance, henries
 L_0 = average value of phase self-inductance, henries
 L_{fd} = direct-axis main-field-winding self-inductance, henries
 l_{oc} = open-circuit leakage inductance, henries
 l_{sc} = short-circuit leakage inductance, henries
 l_B = bucking leakage inductance, henries
 M = mutual inductance, henries

M_0 = amplitude of variation of phase self-inductance and of mutual inductance between phases, henries
 M_d = mutual inductance between the armature and the direct-axis main-field winding, henries
 M_s = average value of mutual inductance between phases, henries
 ϕ = angle between the direct axis and the axis of phase a
 ω = rated angular frequency = $2\pi f$ radians per second
 f = rated frequency, cycles per second
 t = time, seconds
 p = time derivative operator = $\frac{d}{dt}$
 \mathcal{P} = permeance
 x_l = armature leakage reactance
 x_d = direct synchronous reactance
 x_q = quadrature synchronous reactance
 x_d' = direct transient reactance
 x_d'' = direct subtransient reactance
 x_q' = quadrature transient reactance
 x_q'' = quadrature subtransient reactance
 x_2 = negative-sequence reactance
 x_0 = zero-sequence reactance

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