Risoluzione appello del 11/02/2020 (traccia A)

Esercizio 1:

(i) $w = \dfrac{5(2+3i)(-1-2i)}{5} + (-1+4i)$

$= 4-7i + (-1+4i) = 3-3i$

Dunque $\mathrm{Re}(w) = 3$, $\mathrm{Im}(w) = -3$.

(ii) Si ha $X^2 + bX + c = (X-w)(X-(1-w))$,

dunque $X^2+bX+c = X^2 - (w+(1-w))X + w(1-w)$

da cui segue $b = -1$ e

$c = 3(1-i)(-2+3i) = 3+15i$

Esercizio 2: $u = (1,1,2,2,2)$, $v = (3,3,1,1,1)$

$$U = \langle u, v \rangle \subset \mathbb{R}^5 .$$

(i) $U = \langle (1,1,0,0,0), (0,0,1,1,1) \rangle$

Una base ortonormale di U è

$$\left\{ \frac{1}{\sqrt{2}}(1,1,0,0,0), \frac{1}{\sqrt{3}}(0,0,1,1,1) \right\}$$

(ii)

$$U^\perp : \begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

Risolvendo il sistema si trova che una base di $U^\perp$ è ad esempio

$$\{ (1,-1,0,0,0), (0,0,1,-1,0), (0,0,0,1,-1) \}$$

(iii)

$$v_1 = \left((2,0,1,2,0) \cdot \frac{1}{\sqrt{2}}(1,1,0,0,0) \right) \frac{1}{\sqrt{2}}(1,1,0,0,0) +$$

$$+ \left((2,0,1,2,0) \cdot \frac{1}{\sqrt{3}}(0,0,1,1,1) \right) \frac{1}{\sqrt{3}}(0,0,1,1,1)$$

$$= (1,1,0,0,0) + (0,0,1,1,1) = (1,1,1,1,1)$$

e $v_2 = v_1 - (2,0,1,2,0) = (-1,1,0,-1,1)$

Esercizio 3: $\phi_k : \mathbb{R}^3 \to \mathbb{R}^3$

$$(1,0,0) \mapsto (1,1,k)$$
$$(0,1,1) \mapsto (1,1,1)$$
$$(1,0,-1) \mapsto (-1,k,k)$$

(i) Poiché $(0,1,0) = (0,1,1) + (1,0,-1) - (1,0,0)$ e
$(0,0,1) = -(1,0,-1) + (1,0,0)$

segue che
$$\phi_k(0,1,0) = (1,1,1) + (-1,k,k) - (1,1,k)$$
e
$$\phi_k(0,0,1) = (1,-k,-k) + (1,1,k)$$

dunque
$$A_k = A_{\mathcal{E}_3,\mathcal{E}_3,\phi_k} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & k & 1-k \\ k & 1 & 0 \end{pmatrix}$$

(ii) Si ha $\det A_k = k\big((k-1) - 2k\big) - \big((1-k) - 2\big)$

$$= -k(k+1) + (k+1) = (1-k)(k+1)$$

da cui si deduce che

per $k \neq -1, 1$, ϕ_k è invertibile, ossia

$$\ker \phi_k = \{(0,0,0)\}, \quad \operatorname{Im} \phi_k = \mathbb{R}^3.$$

discutiamo ora i casi $k=1$ e $k=-1$.

<u>caso $k=1$:</u> $\quad A_1 = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

si ha $\operatorname{rk} A_1 = 2$

$\operatorname{Im} \phi_1 = \langle (1,0,0), (0,1,1) \rangle$, $\ker \phi_1 = \langle (1,-1,-1) \rangle$

<u>caso $k=-1$:</u> $\quad A_{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$

si ha $\operatorname{rk} A_{-1} = 2$

$\operatorname{Im} \phi_{-1} = \langle (1,1,0), (0,0,1) \rangle$, $\ker \phi_{-1} = \langle (1,1,0) \rangle$

(iii) Si ha $(1,0,0) = (1,1,1) - \frac{1}{k+1}\left[(1,1,k) + (-1,k,1)\right]$

per $k \neq -1$.

Dunque, per $k \neq -1, 1$, poiché ϕ_k è biettiva, $\phi_k^{-1}(1,0,0)$ è costituita da un singolo elemento e si ha

per $k \neq 1,-1$, $\quad \phi_k^{-1}(1,0,0) = \left\{ (0,1,1) - \frac{1}{k+1}(1,1,0) \right\}$

$$= \left\{ \frac{1}{k+1}(-1, k, k+1) \right\}$$

Discutiamo ora i casi $k = 1, -1$.

caso $k=1$: $\quad (1,0,0) \in \operatorname{Im} \phi_1$

$$\phi_1^{-1}(1,0,0) = \left(0,0,\tfrac{1}{2}\right) + \langle (1,-1,-1) \rangle$$

caso $k=-1$: $\quad (1,0,0) \notin \operatorname{Im} \phi_{-1}$

dunque $\phi_{-1}^{-1}(1,0,0) = \emptyset$.

Esercizio 4: $\varphi_k : \mathbb{R}^3 \to \mathbb{R}^3$

$$B_k = A_{\mathcal{E}_3, \mathcal{E}_3, \varphi_k} = \begin{pmatrix} 5 & 3 & -k \\ -6 & -4 & 2k \\ 0 & 0 & -k \end{pmatrix}$$

(i) Polinomio caratteristico di φ_k:

$$P_{\varphi_k}(t) = \det(B_k - tI_3) = \det \begin{pmatrix} 5-t & 3 & -k \\ -6 & -4-t & 2k \\ 0 & 0 & -k-t \end{pmatrix}$$

$$= -(t+k)\left[(5-t)(-4-t) + 18\right] = -(t+k)(t+1)(t-2)$$

caso $k \neq 1, -2$

autovalori $-1, 2, -k$ tutti di molteplicità algebrica 1

caso $k = 1$:

autovalori $-1 \quad m_a(-1) = 2$
$ 2 \quad m_a(2) = 1$

caso $k = -2$:

autovalori $-1 \quad m_a(-1) = 1$
$ 2 \quad m_a(2) = 2$

(ii), (iii) discutiamo i vari casi.

caso $k \neq 1, -2$:

Autospazi :

$$V_{-1} = \ker \begin{pmatrix} 6 & 3 & -k \\ -6 & -3 & 2k \\ 0 & 0 & -k+1 \end{pmatrix} = \langle (1,-2,0) \rangle$$

$$V_2 = \ker \begin{pmatrix} 3 & 3 & -k \\ -6 & -6 & 2k \\ 0 & 0 & -k-2 \end{pmatrix} = \langle (1,-1,0) \rangle$$

$$V_{-k} = \ker \begin{pmatrix} 5+k & 3 & -k \\ -6 & -4+k & 2k \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 5+k & 3 & -k \\ -1+k & -1+k & k \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 4+2k & 2+k & 0 \\ -1+k & -1+k & k \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 2 & 1 & 0 \\ -1+k & -1+k & k \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \langle (k, -2k, k-1) \rangle$$

caso $k=1$:

$$B_1 = \begin{pmatrix} 5 & 3 & -1 \\ -6 & -4 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$V_{-1} = \ker \begin{pmatrix} 6 & 3 & -1 \\ -6 & -3 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \langle (1,-2,0) \rangle \qquad m_g(-1) \neq m_a(-1)$$

$$V_2 = \ker \begin{pmatrix} 3 & 3 & -1 \\ -6 & -6 & 2 \\ 0 & 0 & -3 \end{pmatrix} = \langle (1,-1,0) \rangle$$

ϕ_1 non è diagonalizzabile

caso $k = -2$:

$$B_{-2} = \begin{pmatrix} 5 & 3 & 2 \\ -6 & -4 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$V_{-1} = \ker \begin{pmatrix} 6 & 3 & 2 \\ -6 & -3 & -4 \\ 0 & 0 & 4 \end{pmatrix} = \langle (1,-2,0) \rangle$$

$$V_2 = \ker \begin{pmatrix} 3 & 3 & 2 \\ -6 & -6 & -4 \\ 0 & 0 & 0 \end{pmatrix} = \langle (1,-1,0), (2,0,-3) \rangle$$

diagonalizzabile

caso $k \neq 1, -2$:

diagonalizzabile

$$P_k = \begin{pmatrix} 1 & 1 & k \\ -2 & -1 & -2k \\ 0 & 0 & k-1 \end{pmatrix}, \quad D_k = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -k \end{pmatrix}$$

ϕ_k è diagonalizzabile per ogni $k \neq 1$.

Esercizio 5:

$\pi_1 = (1,0,1) + \langle (2,-1,1), (-1,1,0) \rangle$

$\pi_2 : 3x + y - 2z = 2$

(i) $\pi_1 : x + y - z = 0$

$\pi_2 = (0,2,0) + \langle (1,-3,0), (0,2,1) \rangle$

(ii) $\pi : ax + by + cz = d$

$(a,b,c) = (1,1,-1) \times (3,1,-2) = (-1,-1,-2)$

Imponendo $(1,2,0) \in \pi$ si ottiene

$\pi : x + y + 2z = 3$ o $\pi = (1,2,0) + \langle (1,1,-1), (3,1,-2) \rangle$

(iii) $t = (0,1,0) + \langle (1,-1,1) \rangle$

$\pi_2 : 3x + y - 2z = 2$

t e π_2 sono paralleli poiché $(1,-1,1)$ è soluzione

di $3x + y - 2z = 0$, e dunque

$\mathrm{dist}(t, \pi_2) = \mathrm{dist}((0,1,0), \pi_2) = \frac{1}{\sqrt{14}}$.

Exercício 6: $U, V, W \subset \mathbb{R}^4$

(i) Falso.

Contraexemplo:

$U = \langle (1,0,0,0) \rangle$, $V = \langle (1,1,0,0,0) \rangle$, $W = \langle (0,1,0,0,0) \rangle$

(ii) Falso.

Contraexemplo:

$U = \langle (1,0,0,0), (0,1,0,0) \rangle$, $V = \langle (1,1,1,1) \rangle$

$W = \langle (0,0,1,0), (0,0,0,1) \rangle$.

(iii) Falso.

Contraexemplo:

$U = \langle (1,0,0,0) \rangle$, $V = \langle (1,0,0,0), (0,1,0,0) \rangle$

$W = \langle (0,1,0,0) \rangle$.