ALGEBRA LINEARE E GEOMETRIA

Lezione 3, 06/10/2021

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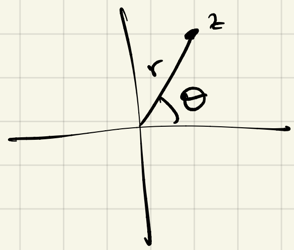
$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$$



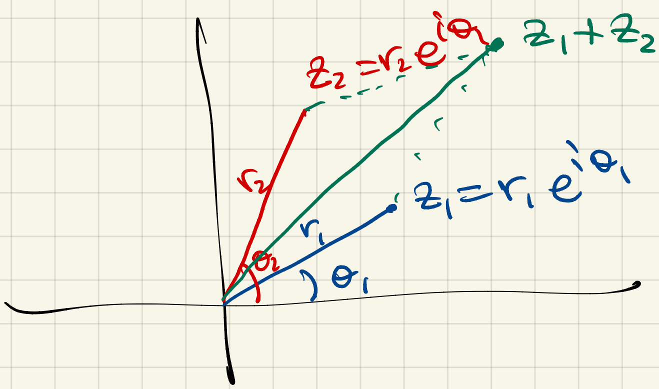
$z = re^{i\theta}$ $\quad$ r = Modulo di z

$r = |z|$

$\theta \in [0, 2\pi)$ $\quad$ $\theta = \mathrm{Arg}(z)$

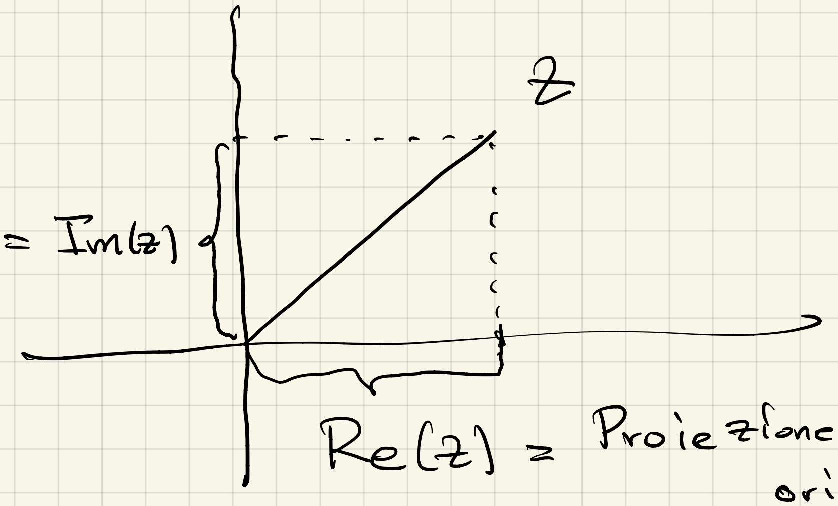
$$\left.\begin{array}{l} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{array}\right\} z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Somma di numeri complessi



È difficile esprimere modulo e argomento di $z_1 + z_2$ in termini di $r_1, r_2, \theta_1, \theta_2$

Prendiamo le proiezioni sugli assi coordinati



Proiezione sull'asse verticale $= \mathrm{Im}(z)$

$\mathrm{Re}(z) =$ Proiezione sull'asse orizzontale

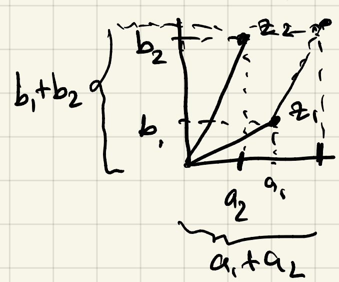
$$\mathrm{Re}(z),\ \mathrm{Im}(z) \in \mathbb{R}$$

$$z = (\mathrm{Re}(z), \mathrm{Im}(z))$$ ← Forma cartesiana di $z \in \mathbb{C}$.

Se

$$z_1 = (a_1, b_1) \quad (\mathrm{Re}(z_1) = a_1,\ \mathrm{Im}(z_1) = b_1)$$

$$z_2 = (a_2, b_2) \quad (\mathrm{Re}(z_2) = a_2,\ \mathrm{Im}(z_2) = b_2)$$



$$\underline{z_1 + z_2} = (a_1 + a_2, b_1 + b_2)$$

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$$

$$\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$$

Somma è associativa

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

Elemento neutro

$$0_{\mathbb{C}} = (0_{\mathbb{R}}, 0_{\mathbb{R}})$$

$$0_{\mathbb{C}} + z = z + 0_{\mathbb{C}} = z \quad \forall z \in \mathbb{C}$$

$$\mathbb{C} = \mathbb{C}^* \cup \{0_{\mathbb{C}}\}$$

Elemento opposto:

$z = (a, b) \in \mathbb{C}$

allora $\underbrace{-z}_{\text{Opposto}} = (-a, -b)$

$$z + (-z) = (a + (-a), b + (-b))$$
$$= (0_{\mathbb{R}}, 0_{\mathbb{R}}) = 0_{\mathbb{C}}$$

Commutativa $\quad z_1 + z_2 = z_2 + z_1$

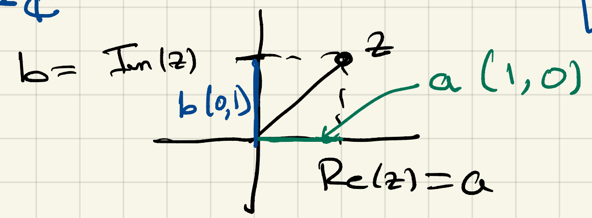
$$\boxed{(\mathbb{C}, +) \text{ è un gruppo}}$$

RAPPRESENTAZIONE ALGEBRICA E TRIGONOMETRICA

Cartesiana

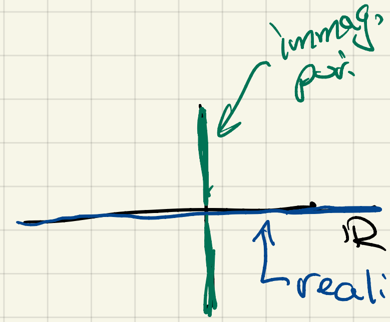
$$z = (a, b) = a \cdot \underbrace{\overset{\boxed{1}}{(1, 0)}}_{1_{\mathbb{C}}} + b \underbrace{(0, 1)}_{\text{unità immaginaria } \boxed{i}}$$

$a = Re(z)$, $b = Im(z)$



$$z = \underbrace{a + ib}_{\text{Rap. algebrica}}$$

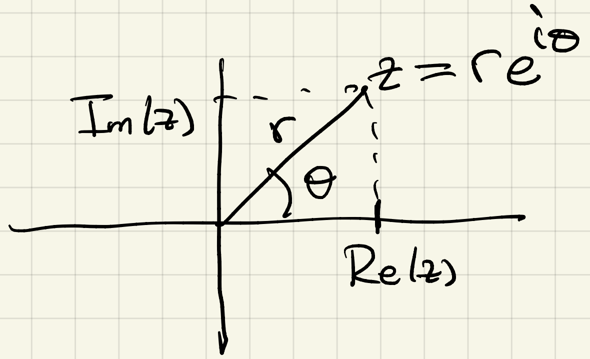
$$\boxed{z = Re(z) + i\, Im(z)}$$



OSS: Se $Im(z) = 0$ diciamo che z è reale

$\mathbb{R} \subset \mathbb{C}$

Se $Re(z) = 0$ diciamo che z è immaginario puro



Quindi:

Pitagora:

$$r = \sqrt{Re(z)^2 + Im(z)^2} = |z|$$

$$\mathrm{Re}(z) = r\cos\theta$$

$$\mathrm{Im}(z) = r\sin\theta$$

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$$

$$= r\cos\theta + i\,r\sin\theta$$

$$= \underbrace{r(\cos\theta + i\sin\theta)}_{\text{Rap. trigonometrica di } z} \qquad r = |z|$$

Esempio

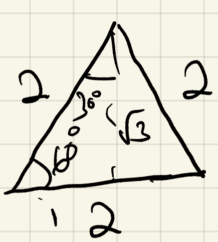
1) $z = \overbrace{(1, -\sqrt{3})}^{\text{Rap. Cartesiana}} = \overbrace{1 - i\sqrt{3}}^{\text{Rap. algebrica}}$

Trovare Rap. esponenziale e trigon.

$$r = |z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$$

Trovare θ: $\begin{cases} \mathrm{Re}(z) = r\cos\theta \\ \mathrm{Im}(z) = r\sin\theta \end{cases} \Rightarrow \begin{matrix} 1 = 2\cos\theta \\ -\sqrt{3} = 2\sin\theta \end{matrix}$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad \sin\theta = -\frac{\sqrt{3}}{2}$$



$$\theta = -\frac{\pi}{3} \quad \text{oppure} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$z = \underbrace{2e^{-i\frac{\pi}{3}} = 2e^{i\frac{5\pi}{3}}}_{\text{Rap. esponenziale.}}$$

$$z = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$$

$$\mathrm{Arg}(z) = \frac{5\pi}{3}.$$

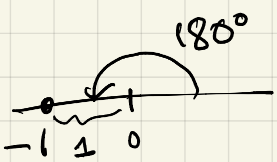
2) $z = -1_{\mathbb{C}} = \underbrace{(-1, 0)}_{\text{cartesiano}} = \underbrace{-1 + i\cdot 0}_{\text{algebrica}}$

$= -1$

Rap. Esponenziale?

$$|z| = \sqrt{(-1)^2 + 0^2} = 1$$

$$\begin{cases} Re(z) = r\cos\theta \\ Im(z) = r\sin\theta \end{cases} \Rightarrow \begin{cases} -1 = \cos\theta \\ 0 = \sin\theta \end{cases}$$



$$\boxed{\theta = \pi} \qquad \boxed{-1 = e^{i\pi}}$$

3) $z = \underbrace{i}_{\text{Rap. algebrica}} = \underbrace{(0, 1)}_{\text{cartesiana}}$

$$|i| = \sqrt{0^2 + 1^2} = 1$$

$|i| = 1.$

$$\begin{cases} Re(z) = r\cos\theta \\ Im(z) = r\sin\theta \end{cases} \qquad \begin{cases} 0 = \cos\theta \\ 1 = \sin\theta \end{cases}$$

$$\theta = \frac{\pi}{2} \qquad i = e^{i\pi/2}$$

$$\sqrt[2]{-1} = \sqrt[2]{e^{i\pi}} \underset{\substack{\uparrow \\ \text{De Moivre}}}{=} \begin{cases} e^{i\pi/2} \\ e^{i(\pi/2+\pi)} = e^{i\frac{3\pi}{2}} \end{cases}$$

$$\underset{\substack{\| \\ i^2}}{\left(e^{i\pi/2}\right)^2} = e^{i\pi} = -1$$

$$\boxed{i^2 = -1}$$

Prodotto con la rapresentazione algebrica:

$$\overset{z_1}{(a+ib)}\overset{z_2}{(c+id)} = (ac - bd) + i(ad + bc)$$

$a, b, c, d \in \mathbb{R}$

Esercizio: Calcolare $\underbrace{(2-2\sqrt{3}i)^4}$

Re? Im?

Modulo? Arg?

$$(2-2\sqrt{3}i)^4 = (2(1-\sqrt{3}i))^4$$

$$= 2^4(1-\sqrt{3}i)^4 \qquad 1-\sqrt{3}i = 2e^{i\frac{5\pi}{3}}$$

$$= 2^4\left(2e^{i\frac{5\pi}{3}}\right)^4$$

$$= 2^4 2^4 e^{i\frac{20\pi}{3}} \qquad \frac{20\pi}{3} = \frac{(18+2)\pi}{3}$$

$$= \underbrace{256\, e^{i\frac{2}{3}\pi}}_{\text{Rap. esp.}} \qquad = 6\pi + \frac{2}{3}\pi$$

$$\mathrm{Arg}\left((2-2\sqrt{3}i)^4\right) = \frac{2}{3}\pi \qquad e^{i\frac{20\pi}{3}} = e^{i\frac{2}{3}\pi}$$

$$|(2-2\sqrt{3}i)^4| = 256$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Allora

$$(2-2\sqrt{3}i)^4 = \overbrace{256\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)}^{\text{Rap. trig.}}$$

$$= \underbrace{-128 + i\,128\sqrt{3}}_{\text{Rap. algebrica}} = \underbrace{(-128,\ 128\sqrt{3})}_{\text{cartesiana}}$$