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ESERCIZIO 1

$\mathbb{R}[x]_{\leq 2}$ $\quad a+bx+cx^2$

$W_1 = \{ a+bx \mid a,b \in \mathbb{R} \}$

$W_2 = \{ a_0 + a_1 x + a_2 x^2 \mid a_0 + a_1 - a_2 = 0 \}$

- Verificare che W_1, W_2 sono sottospazi
- Trovare i generatori per $W_1, W_2, W_1 + W_2$
- Calcolare $W_1 \cap W_2$ e dire se la somma è diretta

RISOLVO

1) $w_1 \in W_1 \quad w_2 \in W_1 \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1 w_1 + \lambda_2 w_1 \in W_1$

$w_1 = a_1 + b_1 x$

$w_2 = a_2 + b_2 x$

$\lambda_1 w_1 + \lambda_2 w_2 = \underline{\lambda_1 a_1} + \underline{\lambda_1 b_1 x} + \underline{\lambda_2 a_2} + \lambda_2 \underline{b_2} x$

$(\lambda_1 a_1 + \lambda_2 a_2) + (\lambda_1 b_1 + \lambda_2 b_2) x \in W_1$

$w_1 \in W_2 \quad w_2 \in W_2 \quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1 w_1 + \lambda_2 w_2 \in W_2$

$w_1 = a_0 + a_1 x + a_2 x^2 \qquad a_0 + a_1 - a_2 = 0$

$w_2 = b_0 + b_1 x + b_2 x^2 \qquad b_0 + b_1 - b_2 = 0$

$\lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 a_0 + \lambda_1 a_1 x + \lambda_1 a_2 x^2 + \lambda_2 b_0 + \lambda_2 b_1 x + \lambda_2 b_2 x^2$

$= \underbrace{(\lambda_1 a_0 + \lambda_2 b_0)} + \underbrace{(\lambda_1 a_1 + \lambda_2 b_1)} x + \underbrace{(\lambda_1 a_2 + \lambda_2 b_2)} x^2 \in W_2$

$$\lambda_1 a_0 + \lambda_2 b_0 + \lambda_1 a_1 + \lambda_2 b_1 - \lambda_1 a_2 - \lambda_2 b_2 \overset{?}{=} 0$$

$$\lambda_1 (\underbrace{a_0 + a_1 - a_2}_{=0}) + \lambda_2 (\underbrace{b_0 + b_1 - b_2}_{=0}) = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0$$

$$W_1 = \{a + bx \mid a, b \in \mathbb{R}\}$$
$$= \{a(1) + b(x) \mid a, b \in \mathbb{R}\}$$
$$= \langle 1, x \rangle$$

$a_2 = a_0 + a_1$

$$W_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0 + a_1 - a_2 = 0\}$$
$$= \{a_0 + a_1 x + a_0 x^2 + a_1 x^2\}$$
$$= \{a_0 (1 + x^2) + a_1 (x + x^2)\}$$
$$= \langle 1 + x^2, x + x^2 \rangle$$

$$W_1 + W_2 = \langle 1, x, 1 + x^2, x + x^2 \rangle$$

$a_0 + a_1 = 0 \rightarrow a_0 = -a_1$

$$W_1 \cap W_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0 + a_1 - a_2 = 0,\ a_2 = 0\}$$
$$= \{-a_1 + a_1 x\}$$
$$= \{a_1(-1 + x)\} = \langle -1 + x \rangle$$

SE UTILIZZIAMO i VETTORI $a + bx + cx^2 \rightsquigarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$W_1 = \left\{\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in \mathbb{R}^3\right\} = \left\{a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}\right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \langle \underbrace{1 + 0x + 0x^2}_{1}, \underbrace{0 + 1x + 0x^2}_{x} \rangle$$

ESERCIZIO 2

$V = \mathbb{R}^5$ Sono in somma diretta?

$W_1 \begin{cases} x_1 - x_3 = 0 \quad (x_1 = x_3) \\ x_4 = 0 \end{cases}$ $\qquad W_2 \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 - x_5 = 0 \end{cases}$

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ 0 \\ x_5 \end{pmatrix} \in \mathbb{R} \right\} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \langle e_1 + e_3, e_2, e_5 \rangle$$

⚠ LA BASE CANONICA DI $\mathbb{R}^5$

$$\Rightarrow e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad e_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 2 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{e_1} + 1 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{e_2} + 0 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{e_3} + 1 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{e_4} + 0 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{e_5}$$

$$= 2e_1 + 1e_2 + 1e_4$$

$$\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 2e_1 - e_2 + e_5 \qquad \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} = -3e_1 + e_3 + 2e_5$$

$$3e_1 - 2e_4 + e_5 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

⚠ LE BASI CANONICHE IN $\mathbb{R}^n$

$\mathbb{R}^2$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\mathbb{R}^3$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

CALCOLIAMO LA DIMENSIONE DI W_1 (extra)

$$W_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \cancel{\lambda_1 = 0} \\ \cancel{0 = 0} \\ \lambda_3 = 0 \end{cases}$$

$\Rightarrow$ LIN INDIP. $\Rightarrow$ DIM $W_1 = 3$

TORNIAMO ALL'ESERCIZIO :

$$W_2 \quad \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 - x_5 = 0 \end{cases}$$

$x_4 = x_5$

$$W_2 = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \\ x_4 \\ x_4 \end{pmatrix} \right\} = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \langle e_1, e_4 + e_5 \rangle$$

$$W_1 + W_2 = \left\langle \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\rangle$$

$$= \langle e_1, e_4+e_5, e_1+e_3, e_2, e_5 \rangle$$

$$W_1 \cap W_2 = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{matrix} x_1 = 0 \\ x_1 = x_3, \; x_4 = 0, \; x_2 = 0, \\ x_3 = 0, \; x_4 = x_5 \\ x_5 = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \right\} = 0_V$$

SOMMA DIRETTA

$\mathbb{R}^3$

$$W_1 = \left\langle \begin{array}{c} a \\ \begin{pmatrix}0\\1\\0\end{pmatrix} \end{array}, \begin{array}{c} b \\ \begin{pmatrix}0\\1\\1\end{pmatrix} \end{array} \right\rangle \qquad W_2 = \left\langle \begin{array}{c} \lambda \\ \begin{pmatrix}1\\0\\0\end{pmatrix} \end{array}, \begin{array}{c} \mu \\ \begin{pmatrix}0\\0\\1\end{pmatrix} \end{array} \right\rangle$$

$W_1 \cap W_2$? ⚠ METODO X QUANDO NON ABBIAMO LE EQ.

$w_1 \in W_1 \qquad w_1 = \begin{pmatrix}0\\a+b\\b\end{pmatrix}$

$w_2 \in W_2 \qquad w_2 = \begin{pmatrix}\lambda\\0\\\mu\end{pmatrix}$

$$w_1 = w_2 \Rightarrow \begin{pmatrix}0\\a+b\\b\end{pmatrix} = \begin{pmatrix}\lambda\\0\\\mu\end{pmatrix} \Rightarrow \begin{cases}\lambda = 0\\ a+b = 0 \\ b = \mu\end{cases}$$

$$\begin{cases}\lambda = 0\\ a = -b \\ b = \mu\end{cases} \qquad \left.\begin{cases}\lambda = 0\\ a = -\mu \\ b = \mu\end{cases}\right] \quad \begin{pmatrix}0\\a+b\\b\end{pmatrix} = \begin{pmatrix}0\\-\mu+\mu\\\mu\end{pmatrix} = \begin{pmatrix}0\\0\\\mu\end{pmatrix}$$

$$W_1 \cap W_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \right\} = \left\{ \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$\mathbb{R}[x]_{\leq 4}$ $\quad p_1(x) = x + x^2 + x^3$ $\quad \dim W_1 = ?$

$p_2(x) = 2 + x^2$

W_1 generato: $p_3(x) = 2x + x^2$

$p_4(x) = x - x^3$

$p_5(x) = 1 - x$

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) + \lambda_4 p_4(x) + \lambda_5 p_5(x) = 0$$

$$\lambda_1 x + \lambda_1 x^2 + \lambda_1 x^3 + \lambda_2 2 + \lambda_2 x^2 + \lambda_3 2x + \lambda_3 x^2 + \lambda_4 x - \lambda_4 x^3 + \lambda_5 - \lambda_5 x = 0$$

$$(2\lambda_2 + \lambda_5) + (\lambda_1 + 2\lambda_3 + \lambda_4 - \lambda_5)x + (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1 - \lambda_4)x^3 = 0$$

$$\begin{cases} 2\lambda_2 + \lambda_5 = 0 \\ \lambda_1 + 2\lambda_3 + \lambda_4 - \lambda_5 = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_1 - \lambda_4 = 0 \end{cases} \qquad \begin{cases} \lambda_5 = -2\lambda_2 \\ \cancel{\lambda_4} - 2\cancel{\lambda_4} - \cancel{2\lambda_2} + \cancel{\lambda_4} + 2\cancel{\lambda_2} = 0 \qquad \cancel{0=0} \\ \lambda_3 = -\lambda_1 - \lambda_2 = -\lambda_4 - \lambda_2 \\ \lambda_1 = \lambda_4 \end{cases}$$

SE $\lambda_2 = 0$ $\quad \begin{cases} \lambda_5 = 0 \\ \lambda_3 = -\lambda_4 \\ \lambda_1 = \lambda_4 \end{cases}$

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) + \lambda_4 p_4(x) + \lambda_5 p_5(x) = 0$$

$$\lambda_4 p_1(x) + 0 - \lambda_4 p_3(x) + \lambda_4 p_4(x) + 0 = 0$$

$$\cancel{\lambda_4} p_1(x) - \cancel{\lambda_4} p_3(x) + \cancel{\lambda_4} p_4(x) = 0$$

$$p_1(x) - p_3(x) + p_4(x) = 0$$

$$p_3(x) = p_1(x) + p_4(x) \qquad \text{SCARTO } p_3(x)$$

SE $\lambda_4 = 0$ $\quad \begin{cases} \lambda_5 = -2\lambda_2 \\ \lambda_3 = -\lambda_2 \end{cases}$

$$\begin{cases} \lambda_3 = -\lambda_2 \\ \lambda_1 = 0 \end{cases}$$

$$\cancel{\lambda_2}\, p_2(x) - \cancel{\lambda_2}\, p_3(x) - 2\cancel{\lambda_2}\, p_5(x) = 0$$

$$p_2(x) = p_3(x) + 2p_5(x)$$ SCARTO $p_2(x)$

$$W_1 = \langle p_1(x),\ p_4(x),\ p_5(x) \rangle$$

$\uparrow$ BASE $\Rightarrow$ $\dim W_1 = 3$