ALGEBRA LINEARE E GEOMETRIA

Lezioni 32-33, 11/11/2021

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Richiamo dalla lezione precedente:

$$f : V = \mathbb{R}^3 \longrightarrow W = \mathbb{R}^2 \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2y + z \\ x + 2z \end{pmatrix}$$

$$\underline{v} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\} \qquad \underline{w} = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$$

Basi di $\mathbb{R}^3$ Basi di $\mathbb{R}^2$

$$\underline{v}' = \left\{ \underset{v_1'}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}, \underset{v_2'}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_3'}{\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}} \right\} \qquad \underline{w}' = \left\{ \underset{w_1'}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}, \underset{w_2'}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \right\}$$

$$A = M^{\underline{w}}_{\underline{v}}(f) = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\hat{A} = M^{\underline{w}}_{\underline{v}'}(f) = \begin{pmatrix} 5 & -4 & 5 \\ 5 & 1 & 6 \end{pmatrix}$$

$$A' = M^{\underline{w}'}_{\underline{v}'}(f) = \begin{pmatrix} 2 & -\frac{3}{5} & \frac{11}{5} \\ -1 & \frac{14}{5} & -\frac{3}{5} \end{pmatrix}$$

MATRICI DI CAMBIAMENTO DI BASE

(vedi libro, Cap. 2, p. 62–67)

V spazio vettoriale su $\mathbb{K}$.

$\underline{v} = \{v_1, \dots, v_n\}$
$\underline{v}' = \{v_1', \dots, v_n'\}$ due basi di V.

$v \in V$
$$v = \lambda_1 v_1 + \dots + \lambda_n v_n = \sum_{i=1}^{n} \lambda_i v_i$$
$$v = \lambda_1' v_1' + \dots + \lambda_n' v_n' = \sum_{i=1}^{n} \lambda_i' v_i'$$

$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}} \in \mathbb{K}^n$ sono le coordinate del vettore v rispetto alla base $\underline{v}$.

$\begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix}^{\underline{v}'} \in \mathbb{K}^n$ sono le coord. del vettore v rispetto alla base $\underline{v}'$.

$$\underset{id(v_i)}{\underset{\shortparallel}{v_i}} = p_{1i} v_1' + p_{2i} v_2' + \dots + p_{ni} v_n' = \sum_{k=1}^{n} p_{ki} v_k'$$

Matrice $P = (p_{ki})$ $\quad k = 1, \dots, n$, $\quad i = 1, \dots, n$

matrice $n \times n$

$$P = \left(\cdots \begin{array}{c} \boxed{\begin{matrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{matrix}} \\ \text{colonna } i \end{array} \cdots \right)$$

Le colonne di P sono le coordinate dei vettori $v_1, \dots, v_n$ rispetto alla 2^a base $v_1', \dots, v_n'$.

P si chiama matrice di cambiamento di base.

Consideriamo la funzione identica

$$id: V \longrightarrow V \qquad id(v) = v \quad \forall v \in V.$$
$$v \longmapsto v$$

Se sul dominio metto la base $\underline{v}$ e sul codominio metto la base $\underline{v}'$ allora si ha

$$\boxed{M_{\underline{v}}^{\underline{v}'}(id) = P}$$

Scambiando il ruolo delle basi, si trova

$$v_k' = q_{1k} v_1 + \dots + q_{nk} v_n = \sum_{l=1}^{n} q_{lk} v_l$$

Matrice $Q = (q_{lk})$ matrice $n \times n$.

$$Q = \left(\cdots \begin{array}{c} \boxed{\begin{matrix} q_{1k} \\ q_{2k} \\ \vdots \\ q_{nk} \end{matrix}} \\ \text{Colonna } k. \end{array} \cdots \right)$$

Le colonne di Q sono le coord. dei vettori della 2^a base $v_1', \dots, v_n'$ rispetto alla 1^a base $v_1, \dots, v_n$

Si ha $\quad Q = M^{\underline{v}}_{\underline{v}'}(id)$

$$v_i = \sum_{k=1}^{n} p_{ki} v_k' = \sum_{k=1}^{n} p_{ki} \sum_{l=1}^{n} q_{lk} v_l$$

$$= \sum_{\substack{k=1,\dots,n \\ l=1,\dots n}} q_{lk} p_{ki} v_l$$

$$= \sum_{l=1}^{n} \underbrace{\left(\sum_{k=1}^{n} q_{lk} p_{ki} \right)}_{\text{Componente } li \text{ di } QP} v_l$$

$$(QP)_{li} = \sum_{k=1}^{n} q_{lk} p_{ki}$$

Ma $\quad v_i = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_n$

$$(QP)_{li} = \begin{cases} 0 & \text{se } l \neq i \\ 1 & \text{se } l = i \end{cases}$$

$$QP = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} = I = \text{matrice identica } n \times n.$$

$QP = I$

Si ha anche $PQ = I$

P e Q sono matrici <u>invertibili</u>.

Scrivo $P = Q^{-1}$, $Q = P^{-1}$.

$$\boxed{\begin{gathered}\left(M^{\underline{v}}_{\underline{v}'}(id)\right)^{-1} = M^{\underline{v}'}_{\underline{v}}(id) \\ M^{\underline{v}}_{\underline{v}'}(id)\, M^{\underline{v}'}_{\underline{v}}(id) = I\end{gathered}}$$

Effetto delle matrici di cambiamento di base sulle coordinate dei vettori

$$v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{n} p_{ki} v_k'$$

$$= \sum_{\substack{i=1,\dots,n \\ k=1,\dots,n}} p_{ki} \lambda_i v_k'$$

$$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} p_{ki} \lambda_i \right) v_k' = \sum_{k=1}^{n} \lambda_k' v_k'$$

$$\Rightarrow \lambda_k' = \sum_{i=1}^{n} p_{ki} \lambda_i$$

$$\begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix} = P \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

Più precisamente $\begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix}^{\underline{v}'} = M^{\underline{v}'}_{\underline{v}}(id) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$

$$f : V \longrightarrow W$$

$\underline{v} = \{v_1, \dots, v_n\}$
$\underline{v}' = \{v_1', \dots, v_n'\}$ basi di V

$\underline{w} = \{w_1, \dots, w_m\}$
$\underline{w}' = \{w_1, \dots, w_m'\}$ basi di W

$A = M_{\underline{v}}^{\underline{w}}(f)$ matrice di f rispetto alle basi $\underline{v}, \underline{w}$.

$A' = M_{\underline{v}'}^{\underline{w}'}(f)$ matrice di f rispetto alle basi $\underline{v}', \underline{w}'$.

$v \in V$

$$v = \sum_{i=1}^{n} \lambda_i v_i \qquad \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$$

$$= \sum_{i=1}^{n} \lambda_i' v_i' \qquad \begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix}^{\underline{v}'}$$

$$\begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix}^{\underline{v}'} = P \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$$

Matrice di cambiamento di base $P = M_{\underline{v}}^{\underline{v}'}(id)$

$w = f(v) \in W$

$$w = \sum_{j=1}^{m} \mu_j w_j = \sum_{j=1}^{m} \mu_j' w_j'$$

$$\begin{pmatrix} \mu_1' \\ \vdots \\ \mu_m' \end{pmatrix}^{\underline{w}'} = \underbrace{S}_{\text{Matrice di cambiamento di base } S = M^{\underline{w}'}_{\underline{w}}(id)} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}^{\underline{w}}$$

Sappiamo

$$\begin{pmatrix} \mu_1' \\ \vdots \\ \mu_m' \end{pmatrix}^{\underline{w}'} = \underbrace{A'}_{M^{\underline{w}'}_{\underline{v}'}(f)} \begin{pmatrix} \lambda_1' \\ \vdots \\ \lambda_n' \end{pmatrix}^{\underline{v}'} = A' \underbrace{P}_{M^{\underline{v}'}_{\underline{v}}(id)} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$$

$$/\!/$$

$$\underbrace{S}_{M^{\underline{w}'}_{\underline{w}}(id)} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}^{\underline{w}} = S \underbrace{A}_{M^{\underline{w}}_{\underline{v}}(f)} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$$

$$\Rightarrow \quad SA \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}} = A'P \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$$

Vero per ogni $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}^{\underline{v}}$

$$\Rightarrow \quad \boxed{\underbrace{S}_{\text{Cambiamento di base in } W}A = A'\underbrace{P}_{\text{Cambiamento di base in } V}} \qquad (1)$$

$$S^{-1}SA = S^{-1}A'P \Rightarrow \boxed{A = S^{-1}A'P} \quad (2)$$

$$SAP^{-1} = A'PP^{-1} \qquad \boxed{A' = SAP^{-1}} \quad (3)$$

Con notazioni più precise, le formule appena viste si scrivono come:

(1) $M^{\underline{w}'}_{\underline{w}}(id)\, M^{\underline{w}}_{\underline{v}}(f) = M^{\underline{w}'}_{\underline{v}'}(f)\, M^{\underline{v}'}_{\underline{v}}(id)$

(2) $M^{\underline{w}}_{\underline{v}}(f) = M^{\underline{w}}_{\underline{w}'}(id) M^{\underline{w}'}_{\underline{v}'}(f) M^{\underline{v}'}_{\underline{v}}(id)$

(3) $M^{\underline{w}'}_{\underline{v}'}(f) = M^{\underline{w}'}_{\underline{w}}(id) M^{\underline{w}}_{\underline{v}}(f) M^{\underline{v}}_{\underline{v}'}(id)$

Problemi da risolvere:

- Stabilire se una matrice quadrata è invertibile.

- Come calcolare l'inversa di una matrice.

Esempio:

$$f : V = \mathbb{R}^3 \longrightarrow W = \mathbb{R}^2 \qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2y + z \\ x + 2z \end{pmatrix}$$

$$\underline{v} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\} \qquad \underline{w} = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$$

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$$\underline{v}' = \left\{ \underset{v_1'}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}, \underset{v_2'}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_3'}{\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}} \right\} \qquad \underline{w}' = \left\{ \underset{w_1'}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}, \underset{w_2'}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} \right\}$$

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Matrici di cambiamento di base:

$$P = M^{\underline{v}'}_{\underline{v}}(id) = ?$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$$

$\Rightarrow$ Sistema di eq. lineari per $\alpha_1, \alpha_2, \alpha_3$. Risolvo.

1ª colonna di P... complicato.

$$P^{-1} = M^{\underline{v}}_{\underline{v}'}(id) \qquad \begin{pmatrix}1\\0\\2\end{pmatrix} = 1 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + 0 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + 2 \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\begin{pmatrix}1\\0\\2\end{pmatrix}$ 1^a colonna di P^{-1}

2^a colonna di P^{-1} è $\begin{pmatrix}-1\\1\\1\end{pmatrix} = -1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$

3^a colonna di P^{-1} è $\begin{pmatrix}0\\-1\\3\end{pmatrix}$

$$P^{-1} = M^{\underline{v}}_{\underline{v}'}(id) = \begin{pmatrix}1 & -1 & 0\\0 & 1 & -1\\2 & 1 & 3\end{pmatrix}$$

$$S^{-1} = M^{\underline{w}}_{\underline{w}'}(id) = \begin{pmatrix}2 & -1\\3 & 1\end{pmatrix}$$

$$S = M^{\underline{w}'}_{\underline{w}}(id) \qquad \begin{pmatrix}1\\0\end{pmatrix} = \alpha_1\begin{pmatrix}2\\3\end{pmatrix} + \alpha_2\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\begin{cases}2\alpha_1 - \alpha_2 = 1\\3\alpha_1 + \alpha_2 = 0\end{cases} \qquad \begin{cases}5\alpha_1 = 1\\\alpha_2 = -3\alpha_1\end{cases}$$

$$\begin{cases}\alpha_1 = \frac{1}{5}\\\alpha_2 = -\frac{3}{5}\end{cases}$$

1^a colonna di S è $\begin{pmatrix}\frac{1}{5}\\-\frac{3}{5}\end{pmatrix}$

$$\begin{pmatrix}0\\1\end{pmatrix} = \alpha_1\begin{pmatrix}2\\3\end{pmatrix} + \alpha_2\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\begin{cases}2\alpha_1 - \alpha_2 = 0\\3\alpha_1 + \alpha_2 = 1\end{cases} \qquad \begin{cases}\alpha_2 = 2\alpha_1\\5\alpha_1 = 1\end{cases} \qquad \begin{cases}\alpha_2 = \frac{2}{5}\\\alpha_1 = \frac{1}{5}\end{cases}$$

2^a colonna di S è $\begin{pmatrix}\frac{1}{5}\\\frac{2}{5}\end{pmatrix}$

$$\Rightarrow \quad S = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

(Verificare che $\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$)

La formula (3) ci dice: $\boxed{A' = S A P^{-1}}$

$$\hat{A} = A \underbrace{P^{-1}}_{\text{Camb. di base in } V} \qquad\qquad A' = \underbrace{S}_{\text{Camb. di base in } W} A \underbrace{P^{-1}}_{\text{Camb. di base in } V}$$

$$\underbrace{\begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}}_{P^{-1}} = \underbrace{\begin{pmatrix} 5 & -4 & 5 \\ 5 & 1 & 6 \end{pmatrix}}_{\hat{A}\ \checkmark}$$

$$\underbrace{\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix}}_{P^{-1}} =$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 5 & -4 & 5 \\ 5 & 1 & 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -\frac{3}{5} & \frac{11}{5} \\ -1 & \frac{14}{5} & -\frac{3}{5} \end{pmatrix}}_{A'\ \checkmark}$$