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RANGO DI MATRICE

$$A = \begin{pmatrix} 1 & 0 & 1 & k \\ k & k & 2+k & k^2+k \\ -1 & k & k & 0 \\ 0 & -k^2 & -k-1 & -1 \end{pmatrix}$$

determina $rk(A)$ al variare di $k \in \mathbb{R}$

$$\xrightarrow[\mathrm{III}+\mathrm{I}]{\mathrm{II}-k\mathrm{I}} \begin{pmatrix} 1 & 0 & 1 & k \\ 0 & k & 2 & k \\ 0 & k & k+1 & k \\ 0 & -k^2 & -k-1 & -1 \end{pmatrix} \xrightarrow[\mathrm{IV}+k\mathrm{II}]{\mathrm{III}-\mathrm{II}} \begin{pmatrix} 1 & 0 & 1 & k \\ 0 & k & 2 & k \\ 0 & 0 & k-1 & 0 \\ 0 & 0 & k-1 & k^2-1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{IV}-\mathrm{III}} \begin{pmatrix} 1 & 0 & 1 & k \\ 0 & k & 2 & k \\ 0 & 0 & k-1 & 0 \\ 0 & 0 & 0 & k^2-1 \end{pmatrix}$$

Se $k \neq 0$ e $k \neq \pm 1$ allora $rkA = 4$

Se $k=0$: $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$rkA = 3$

$$\xrightarrow{\mathrm{III}+\frac{1}{2}\mathrm{II}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{\mathrm{III}\leftrightarrow\mathrm{IV}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Se $K=+1$: $A=\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\text{rk } A = 2$

Se $K=-1$: $A=\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\text{rk } A = 3$

- $A = \begin{pmatrix} 0 & 0 & a-a^3 & a^2 & 3a-1 \\ a & 1-a & 0 & 0 & 1 \\ a^2 & a-a^2 & 1-a^2 & a & 2+a \\ 0 & 0 & a^2-1 & -2 & -a \end{pmatrix}$ $\times$

Determina il rango al variare di $a \in \mathbb{R}$

$a \neq 0$, $a \neq \pm 1$, $a \neq 2$, $\text{rk} A = 4$

$a = 0$: $\text{rk} A = 4$

$a = 1$: $\text{rk} A = 3$

$a = -1$: $\text{rk} A = 3$

$a = 2$: $\text{rk} A = 3$

SISTEMI LINEARI

V, $\dim V = n$

$\mathbb{R}^4$ $S: \{ x_1 + 2x_2 = 2$

$\mathbb{R}^4$ $S: \begin{cases} x_1 + 2x_2 = 2 \\ x_3 + x_4 = -1 \\ x_1 + 2x_2 + 2x_3 + 2x_4 = 1 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & 2 & 1 \end{array}\right) = (A|b)$$

NB) $rk\,A \overset{?}{=} rk\,A|b$ $\longrightarrow$ NO : IMPOSSIBILE

$\searrow$ SÍ

$rk\,A = n$ DETERMINATO

$rk\,A < n$ INDETERMINATO

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{III - I} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -1 \end{array}\right) \xrightarrow{III - 2II} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

IL SIST E' IMPOSSIBILE

$rk(A|b) = 3 \neq rk(A) = 2$

$S: \begin{cases} x_1 - x_3 = 4 \\ 2x_2 - 2x_3 = 5 \\ x_1 + 3x_3 = 6 \end{cases}$ $V = \mathbb{R}^3$ $n = 3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 2 & -2 & 5 \\ 1 & 0 & 3 & 6 \end{array}\right) \xrightarrow{III-I} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 4 & 2 \end{array}\right)$$

$rk\,A = 3$
$rk(A|b) = 3$

$$\begin{cases} x_1 - x_3 = 4 \\ 2x_2 - 2x_3 = 5 \\ 4x_3 = 2 \end{cases} \qquad \begin{cases} * \\ 2x_2 - 1 = 5 \\ x_3 = 1/2 \end{cases} \qquad \begin{cases} x_1 - 1/2 = 4 \\ x_2 = 3 \\ x_3 = 1/2 \end{cases}$$

$$\begin{cases} x_1 = 9/2 \\ x_2 = 3 \\ x_3 = 1/2 \end{cases} \qquad Sol = \begin{pmatrix} 9/2 \\ 3 \\ 1/2 \end{pmatrix}$$

- $$\begin{cases} x_1 - 2x_2 = 5 \\ x_1 - 3x_3 = -2 \\ x_2 + x_4 = 3 \\ 2x_1 - 2x_2 - 3x_3 = 3 \end{cases}$$ $\mathbb{R}^4$ $\quad n = 4$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \\ 1 & 0 & -3 & 0 & -2 \\ 0 & 1 & 0 & 1 & 3 \\ 2 & -2 & -3 & 0 & 3 \end{array}\right) \xrightarrow[IV-2I]{II-I} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \\ 0 & 2 & -3 & 0 & -7 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 2 & -3 & 0 & -7 \end{array}\right)$$

$$\xrightarrow{IV-II} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \\ 0 & 2 & -3 & 0 & -7 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{II \leftrightarrow III} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 2 & -3 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{III-2II} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \end{array}\right.$$

$rk\,A = rk(A|b) = 3 < 4$

$$III - 2II \rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$rk A = rk(A|b) = 3 < 4$

INDET.

$$\begin{cases} x_1 - 2x_2 = 5 \\ x_2 + x_4 = 3 \\ +3x_3 + 2x_4 = +13 \end{cases} \qquad \begin{cases} x_1 + 2x_4 - 6 = 5 \\ x_2 = -x_4 + 3 \\ x_3 = -\frac{2}{3}x_4 + \frac{13}{3} \end{cases} \qquad \begin{cases} x_1 = -2x_4 + 11 \\ x_2 = -x_4 + 3 \\ x_3 = -\frac{2}{3}x_4 + \frac{13}{3} \end{cases}$$

$$Sol = \begin{pmatrix} 11 \\ 3 \\ 13/3 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -2 \\ -1 \\ -2/3 \\ 1 \end{pmatrix} \right\rangle$$

AUTOVALORI/AUTOVETTORI

- $A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \qquad A - \lambda \mathbb{1} = \begin{pmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{pmatrix}$

$$p_A(x) = \det \begin{pmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) - 3 =$$

$$= -2\lambda + \lambda^2 - 3 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\lambda_1 = -1 \qquad \lambda_2 = 3$

$m_a(-1) = 1 \qquad m_a(3) = 1$

$m_g(-1) = 1 \qquad m_g(3) = 1$

$$A - \lambda_1 \mathbb{1} = A + \mathbb{1} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\begin{cases} 3x_1 + x_2 = 0 \\ 3x_1 + x_2 = 0 \end{cases} \quad \begin{cases} x_2 = -3x_1 \\ 0 = 0 \end{cases} \quad V_{-1} = \left\langle \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\rangle$$

$$A - \lambda_2 \mathbb{1} = A - 3\mathbb{1} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right) \xrightarrow{II + 3I} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-x_1 + x_2 = 0 \quad ; \quad x_1 = x_2 \qquad V_3 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

$\Rightarrow$ A e' DIAGONALIZZABILE

$$D = P^{-1} A P \quad \rightsquigarrow^{\text{OPPURE}} \quad PD = AP$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$$

$$P^{-1}: \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{II + 3I} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{4} II} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ \end{array}\right)$$

$$\xrightarrow{\div 4\ \text{II}} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/4 & 1/4 \end{array}\right)$$

$$\xrightarrow{\text{I}-\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & 1/4 & -1/4 \\ 0 & 1 & 3/4 & 1/4 \end{array}\right) \qquad P^{-1} = \begin{pmatrix} 1/4 & -1/4 \\ 3/4 & 1/4 \end{pmatrix}$$

- $$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$p_A(x) = \det\begin{pmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} =$$

$$= +(3-\lambda)\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)\left[(1-\lambda)(3-\lambda)+1\right]$$

$$= (3-\lambda)(3-3\lambda-\lambda+\lambda^2+1)$$

$$= (3-\lambda)(\lambda^2-4\lambda+4) = (3-\lambda)(\lambda-2)^2$$

AUTOVALORI SONO i λ PER CUI $p_A(x)=0$

$$(3-\lambda)^1(\lambda-2)^2 = 0$$

- $3-\lambda=0 \to \lambda=3$
- $(\lambda-2)^2=0 \to \lambda-2=0 \to \lambda=2$

$\lambda_1 = 2$ $\qquad$ $\lambda_2 = 3$

$m_a(\lambda_1) = 2$ $\qquad$ $m_a(\lambda_2) = 1$

$m_g(\lambda_2) = ?$ $\qquad$ $m_g(\lambda_2) = 1$

CALCOLIAMO GLI AUTOSPAZI