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$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$p_A(\lambda) = (3-\lambda)(\lambda-2)^2$$

$\lambda_1 = 2$ $\qquad$ $\lambda_2 = 3$

$ma(2) = 2$ $\qquad$ $ma(3) = mg(3) = 1$

$$A - 2\mathbb{1} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} -x_1 - x_2 - x_3 = 0 \\ x_1 + x_2 = 0 \\ x_3 = 0 \end{cases} \qquad \begin{cases} x_2 - x_2 - 0 = 0 \\ x_1 = -x_2 \\ x_3 = 0 \end{cases} \qquad 0 = 0$$

$$V_2 = \left\{ \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \right\} = \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$mg(2) = 1 \neq ma(2)$ $\qquad$ A NON É DIAGON.

$$A - 3\mathbb{1} = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\mathrm{I} \leftrightarrow \mathrm{II}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\mathrm{II} + 2\mathrm{I}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x_1 = 0 \\ -x_2 - x_3 = 0 \end{cases} ; \quad x_2 = -x_3$$

$$V_3 = \left\{ \begin{pmatrix} 0 \\ -x_3 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \right\} = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

1) POLINOMIO $\det(A - \lambda \mathbb{1})$

2) SCOMPONETE $\longrightarrow$ AUTOVALORI λ_i

3) M. ALG.

4) AUTOSPAZI . $\operatorname{Ker}(A - \lambda_i \mathbb{1})$

λ_i ↑ AUTOVALORE

↳ AUTOVETTORI

5) M. GEOM

6) $m_A = m_G \quad \forall \lambda_i$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad P = \begin{pmatrix} v_1 & \dots & v_n \\ \vdots & & \end{pmatrix}$$

7) P^{-1} / $\quad AP = PD \qquad (D = P^{-1}AP)$

ESERCIZIO

$$f: \mathbb{R}[x]_{\leq 2} \longrightarrow \mathbb{R}[x]_{\leq 2}$$

$$a_0 + a_1x + a_2x^2 \longmapsto (a_0 + a_1 - a_2) + (a_0 + 3a_1)x + (-7a_0 + 3a_2)x^2$$

Dare una base per il dominio e il codominio per cui M associata a f è DIAGONALE

$\mathcal{E} = \{1, x, x^2\} \qquad A = M_{\mathcal{E}}^{\mathcal{E}}(f)$

$f(1) = 1 + x - 7x^2 \longrightarrow (1, 1, -7)$

$1 = 1 + 0x + 0x^2$ ovvero $a_0 = 1 \quad a_1 = a_2 = 0$

$f(x) = 1 + 3x \longrightarrow (1, 3, 0)$

$f(x^2) = -1 + 3x^2 \longrightarrow (-1, 0, 3)$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & 0 \\ -7 & 0 & 3 \end{pmatrix}$$

$$p_A(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & -1 \\ 1 & 3-\lambda & 0 \\ -7 & 0 & 3-\lambda \end{pmatrix}$$

$$= -7\det\begin{pmatrix} 1 & -1 \\ 3-\lambda & 0 \end{pmatrix} + (3-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= -7\left(0 + 3-\lambda\right) + (3-\lambda)\left[(1-\lambda)(3-\lambda) - 1\right]$$

$$= (3-\lambda)\left[-7 + 3 - 3\lambda - \lambda + \lambda^2 - 1\right]$$

$$= (3-\lambda)(\lambda^2 - 4\lambda - 5)$$

$$= (3-\lambda)(\lambda-5)(\lambda+1)$$

$\lambda_1 = -1$ $\qquad\qquad$ $\lambda_2 = 3$ $\qquad\qquad$ $\lambda_3 = 5$

$m_a(\lambda_1) = m_g(\lambda_1) = 1$ $\qquad$ $m_a(\lambda_2) = m_g(\lambda_2) = 1$ $\qquad$ $m_a(\lambda_3) = m_g(\lambda_3) = 1$

$$A + \mathbb{1} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 0 \\ -7 & 0 & 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 4 & 0 & 0 \\ -7 & 0 & 4 & 0 \end{array}\right) \xrightarrow{I \leftrightarrow II} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ -7 & 0 & 4 & 0 \end{array}\right)$$

$$\xrightarrow[III+7I]{II-2I} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right) \xrightarrow{III+4II} \left(\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{III+7I} \left(\begin{array}{ccc|c} 0 & -7 & -1 & 0 \\ 0 & 28 & 4 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 4x_2 = 0 \\ -7x_2 - x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = -4x_2 \\ x_3 = -7x_2 \end{cases}$$

$$V_{-1} = \left\{ \begin{pmatrix} -4x_2 \\ x_2 \\ -7x_2 \end{pmatrix} \in \mathbb{R}^3 \right\} = \left\langle \begin{pmatrix} -4 \\ 1 \\ -7 \end{pmatrix} \right\rangle$$

$$A - 3\mathbb{1} = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ -7 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} -2x_1 + x_2 - x_3 = 0 \\ x_1 = 0 \\ -7x_1 = 0 \end{cases} \qquad \begin{cases} x_2 = x_3 \\ x_1 = 0 \\ x_1 = 0 \end{cases} \qquad V_3 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$A - 5\mathbb{1} = \begin{pmatrix} -4 & 1 & -1 \\ 1 & -2 & 0 \\ -7 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -4 & 1 & -1 & 0 \\ 1 & -2 & 0 & 0 \\ -7 & 0 & -2 & 0 \end{array}\right) \xrightarrow{I \leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -4 & 1 & -1 & 0 \\ -7 & 0 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[III+7I]{II+4I} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -14 & -2 & 0 \end{array}\right) \xrightarrow{III-2II} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$III + 7I \quad \left(\begin{array}{ccc|c} 0 & -7 & -1 & 0 \\ 0 & -14 & -2 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - 2x_2 = 0 \\ -7x_2 - x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = 2x_2 \\ x_3 = -7x_2 \end{cases}$$

$$V_5 = \left\langle \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} \right\rangle$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad P = \begin{pmatrix} -4 & 0 & 2 \\ 1 & 1 & 1 \\ -7 & 1 & -7 \end{pmatrix}$$

$$B = \{ -4 + x - 7x^2 ;\ x + x^2 ;\ 2 + x - 7x^2 \}$$

$$\begin{pmatrix} -4 \\ 1 \\ -7 \end{pmatrix} \rightsquigarrow -4 + x - 7x^2$$

$$\mathbb{R}^3 = U \oplus W$$

$$U = \langle \underbrace{e_1}_{u_1},\ \underbrace{e_1 + e_2}_{u_2} \rangle \qquad W = \langle \underbrace{e_2 + e_3}_{w_1} \rangle$$

Si considere $\pi_U^W : \mathbb{R}^3 \longrightarrow U$

$$v = u + w \longmapsto u$$

1) CALCOLARE $\pi_U^W(v)$ $\quad v = \begin{pmatrix} 3 \end{pmatrix}$

1) CALCOLARE $\pi_U^W(v)$ $\quad v = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

METODO 1: "SCOMPONGO" $v = u + w$

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 w_1$$

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} 3 = \lambda_1 + \lambda_2 \\ -1 = \lambda_2 + \lambda_3 \\ 1 = \lambda_3 \end{cases} \qquad \begin{cases} \lambda_1 = 5 \\ \lambda_2 = -2 \\ \lambda_3 = 1 \end{cases}$$

$$v = \underbrace{5u_1 - 2u_2}_{u} + \underbrace{w_1}_{w} = u + w$$

$$\pi_U^W(v) = 5u_1 - 2u_2 = 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$

METODO 2: COSTRUISCO LA MATRICE

a) $\qquad$ b) $\circledast$

↳ FACCIO I CONTI PER e_1, e_2, e_3

$$e_1 = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 w_1$$

$$\begin{cases} 1 = \lambda_1 + \lambda_2 + 0 \\ 0 = 0 + \lambda_2 + \lambda_3 \\ 0 = 0 + 0 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases}$$

$$e_1 = 1u_1 + 0u_2 + 0w_1 \qquad \pi_U^W(e_1) = 1u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e_2 = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 w_1$$

$$\begin{cases} 0 = \lambda_1 + \lambda_2 + 0 \\ 1 = 0 + \lambda_2 + \lambda_3 \\ 0 = 0 + 0 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \end{cases}$$

$$e_2 = -1u_1 + 1u_2 + 0w_1 \qquad \pi_U^W(e_2) = -u_1 + u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_3 = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 w_1$$

$$\begin{cases} 0 = \lambda_1 + \lambda_2 + 0 \\ 0 = 0 + \lambda_2 + \lambda_3 \\ 1 = 0 + 0 + \lambda_3 \end{cases} \qquad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \\ \lambda_3 = 1 \end{cases}$$

$$e_3 = u_1 - u_2 + w_1 \qquad \pi_U^W(e_3) = u_1 - u_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\pi_U^W(v) = Av = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} = \begin{pmatrix} 3 \end{pmatrix}$$

$$\pi_U^W(v) = A \cdot v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$$

(*) USO LA FORMA DIAGONALE

$$B = \{ u_1, u_2, w_1 \}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = P \cdot D \cdot P^{-1}$$

$$P^{-1}: \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{II - III} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{I - II} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad P^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = PDP^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = A\,!$$

$\pi_U^W(v) = A \cdot v$ come prima

$$A = \begin{pmatrix} K & 1 & 0 \\ -4 & K-5 & K-3 \\ 0 & 0 & -2 \end{pmatrix} \qquad K \in \mathbb{R}$$

1) Studiare per quali $K \in \mathbb{R}$ si ha $\lambda = -3$ autoval

$\lambda = -3$ è autoval $\iff$ $\det(A + 3\mathbb{1}) = 0$

(viene dal fatto che $p_A(\lambda) = \det(A - \lambda\mathbb{1})$)

$$A + 3\mathbb{1} = \begin{pmatrix} K+3 & 1 & 0 \\ -4 & K-2 & K-3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A+3\mathbb{1}) = 1\det\begin{pmatrix} k+3 & 1 \\ -4 & k-2 \end{pmatrix} =$$

$$= (k+3)(k-2)+4$$

$$= k^2+3k-2k-6+4$$

$$= k^2+k-2 = (k+2)(k-1)$$

$$\det(A+3\mathbb{1}) = 0 \iff k=-2 \vee k=1$$

2) Studiare per quali valori di $k \in \mathbb{R}$ si ha $v = e_1+e_3$ sia autovettore

v è autovettore $\iff Av = \lambda v \quad \exists \lambda \in \mathbb{R}$

$$Av = \begin{pmatrix} k & 1 & 0 \\ -4 & k-5 & k-3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k-7 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} k \\ k-7 \\ -2 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} k=\lambda \\ k-7=0 \\ -2=\lambda \end{cases} \quad \begin{cases} k=\lambda \text{ ASSURDO} \\ k=7 \\ \lambda=-2 \end{cases}$$

$\nexists k \in \mathbb{R}$ tc v è autovettore

3) Studiare $k \in \mathbb{R}$ tc $u = e_3$ sia autovettore

$$Au = \begin{pmatrix} K & 1 & 0 \\ -4 & K-5 & K-3 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ K-3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ K-3 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 0 = 0 \\ K-3 = 0 \\ -2 = \lambda \end{cases} \quad \begin{cases} 0 = 0 \\ K = 3 \\ \lambda = -2 \end{cases}$$

Se $K=3$ allora u e' autovettore di autovalore $\lambda = -2$.

4) Se e' possibile, diagonalizzare A nel caso di $K=1$