SIMULAZIONE 1144237

ESERCIZIO 1

$$A_a = \begin{pmatrix} 2 & a & a & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 2 & a & a & -1 \end{pmatrix}$$

a) Calcolare il determinante di A_7

$$A_7 = \begin{pmatrix} 2 & 7 & 7 & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 2 & 7 & 7 & -1 \end{pmatrix}$$

$$|A_7| = 2 \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 2 \\ 7 & 7 & -1 \end{vmatrix} - \begin{vmatrix} 7 & 7 & 2 \\ 1 & 2 & 2 \\ 7 & 7 & -1 \end{vmatrix} - \begin{vmatrix} 7 & 7 & 2 \\ 2 & 1 & 3 \\ 7 & 7 & -1 \end{vmatrix} - 2 \begin{vmatrix} 7 & 7 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= 2\left[2 \begin{vmatrix} 2 & 2 \\ 7 & -1 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix} + 7 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}\right] +$$

$$-1\left[7 \begin{vmatrix} 2 & 2 \\ 7 & -1 \end{vmatrix} - \begin{vmatrix} 7 & 2 \\ 7 & -1 \end{vmatrix} + 7 \begin{vmatrix} 7 & 2 \\ 2 & 2 \end{vmatrix}\right] +$$

$$-1\left[7 \begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix} - 2 \begin{vmatrix} 7 & 2 \\ 7 & -1 \end{vmatrix} + 7 \begin{vmatrix} 7 & 2 \\ 1 & 3 \end{vmatrix}\right] +$$

$$-2\left[7 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 7 & 2 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 7 & 2 \\ 1 & 3 \end{vmatrix}\right] =$$

$$= 2\left(2(-2-14) - (-1-21) + 7(2-6)\right)$$
$$- \left(7(-2-14) - (-7-14) + 7(14-4)\right)$$
$$- \left(7(-1-21) - 2(-7-14) + 7(21-2)\right)$$
$$- 2\left(7(2-6) - 2(14-4) + (21-2)\right)$$

$$= -3(-2-14) - 9(-1-21) + 3(-7-14) - 3(14-4)$$
$$- 9(21-2)$$

$$= 3(16 + 66 - 21 - 10 - 57) = 3(82 - 88) = -18$$

→ Porto a scala la matrice

Attenzione che gli scambi comportano un cambio di segno!

b) Si pone $a=0$. Determinare gli autovalori di A_0, le m_a, m_g, gli autospazi. Diagonalizzare A_0 se possibile

$$A_0 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & -1 \end{pmatrix}$$

$$p_{A_0}(x) = \begin{vmatrix} 2-x & 0 & 0 & 2 \\ 1 & 2-x & 1 & 3 \\ -1 & 1 & 2-x & 2 \\ 2 & 0 & 0 & -1-x \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & 0 & 2 \\ 2-x & 1 & 3 \\ 1 & 2-x & 2 \end{vmatrix} + (-1-x) \begin{vmatrix} 2-x & 0 & 0 \\ 1 & 2-x & 1 \\ -1 & 1 & 2-x \end{vmatrix}$$

$$= -2(2) \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} + (-1-x)(2-x) \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix}$$

$$= \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix} \left(-4 + (-1-x)(2-x)\right)$$

$$= \left[(2-x)^2 - 1\right]\left[-4 - 2 + x - 2x + x^2\right]$$

$$= \left((2-x)-1\right)\left((2-x)+1\right)\left(x^2 - x - 6\right)$$

$$= (1-x)(3-x)(x-3)(x+2) = +(x-1)(x-3)^2(x+2)$$

$\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = -2$

$m_a(1) = 1 = m_g(1) \qquad m_a(-2) = 1 = m_g(-2) \qquad m_a(3) = 2$

V_3: $\operatorname{Ker}(A_0 - 3\mathbb{1}_4)$

$$\left(\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 0 \\ 1 & -1 & 1 & 3 & 0 \\ -1 & 1 & -1 & 2 & 0 \\ 2 & 0 & 0 & -4 & 0 \end{array}\right) \xrightarrow[\substack{III - I \\ IV + 2I}]{II + I} \left(\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 5 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{III + II} \left(\begin{array}{cccc|c} -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} -a + 2d = 0 \\ -b + c + 5d = 0 \\ 5d = 0 \end{cases}$$

$$\begin{cases} a = 0 \\ b = c \\ d = 0 \end{cases} \qquad V_3 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad m_g(3) = 1$$

$\Rightarrow A_0$ NON DIAG

Scansionato con CamScanner

V_1. $\mathrm{Ker}(A - \mathbb{1}_4)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow[\substack{III+I \\ IV-2I}]{II-I} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{III-II} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{IV+\frac{2}{3}III} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} a+2d=0 \\ b+c+d=0 \\ 3d=0 \end{cases} \quad \begin{cases} a=0 \\ b=-c \\ d=0 \end{cases} \quad V_1 = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

V_{-2}: $\mathrm{Ker}(A+2\mathbb{1}_4)$

$$\left(\begin{array}{cccc|c} 4 & 0 & 0 & 2 & 0 \\ 1 & 4 & 1 & 3 & 0 \\ -1 & 1 & 4 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{I \leftrightarrow II} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 4 & 0 & 0 & 2 & 0 \\ -1 & 1 & 4 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{III+I \\ IV-2I}]{II-4I} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & -16 & -4 & -10 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & -8 & -2 & -5 & 0 \end{array}\right) \xrightarrow[\substack{\frac{1}{5}III \\ II \leftrightarrow III}]{-\frac{1}{2}II} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & +8 & 2 & 5 & 0 \\ 0 & -8 & -2 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[III-8II]{+III} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{cases} a+4b+c+3d=0 \\ b+c+d=0 \\ 2c+d=0 \end{cases}$$

$$\begin{cases} a+4b+c-6c=0 \\ b+c-2c=0 \\ d=-2c \end{cases} \quad \begin{cases} a+4c+c-6c=0 \\ b=c \\ d=-2c \end{cases} \quad \begin{cases} a=c \\ b=c \\ d=-2c \end{cases} \quad V_{-2} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ -2 \end{pmatrix} \right\rangle$$

c) Sia f l'endomorfismo di $\mathbb{R}^4$ avente matrice A_0 rispetto alla base canonica. Se ne determinino nucleo e immagine e si scriva la matrice di f rispetto alla base $V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$f: \mathbb{R}^4 \to \mathbb{R}^4$ di matrice A_0

$Ker(f) = Ker(A_0) \Rightarrow \underline{x} \text{ t.c. } Ax = 0$

$$\left(\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ -1 & 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow[\substack{III + 1/2\, I \\ IV - I}]{II - 1/2\, I} \left(\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{III - 1/2\, II} \left(\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 3/2 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right) \qquad Ker(f) = 0_V$$

$Im(f) = \langle f(e_1), f(e_2), f(e_3), f(e_4) \rangle$

$$f(e_1) = A_0 e_1 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 2 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$$Im(f) = \left\langle \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle$$

$$f(e_1 + e_2) = f(e_1) + f(e_2) = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 2 \end{pmatrix} = 3v_1 - v_3 + 2v_4$$

$$f(e_3 - e_4) = f(e_3) - f(e_4) = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} = -2v_1 + v_4$$

$$f(e_1) = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix} = v_1 - v_2 + v_3 + v_4$$

$$f(e_4) = \begin{pmatrix} 2 \\ 3 \\ 2 \\ -1 \end{pmatrix} = 3v_1 + 2v_2 - v_3 + v_4$$

$$M_{VV}(f) = M_V^V(f) = \begin{pmatrix} 3 & -2 & 1 & 3 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

$\longrightarrow$ Oppure con matrici del cambiamento di base

d) Gli autovalori di A_0 sono autovalori per A_3

Sfruttiamo il fatto che se λ è autovalore allora $\dim(\ker(A-\lambda I_n)) \geq 1$

$$A_3 = \begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 2 & 3 & 3 & -1 \end{pmatrix} \qquad \lambda=1 \quad \lambda=3 \quad \lambda=-2$$

$\lambda=1$:

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ 2 & 3 & 3 & -2 & 0 \end{array}\right) \xrightarrow[\substack{III+I \\ IV-2I}]{II-I} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & -2 & -2 & 1 & 0 \\ 0 & 4 & 4 & 4 & 0 \\ 0 & -3 & -3 & -6 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{-1/3\, IV \\ II \leftrightarrow III}]{1/4\, III} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow[IV-II]{III+2II} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{IV-1/3\, III} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\lambda=1$ è autovalore

$\lambda=3$:

$$\left(\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 1 & -1 & 1 & 3 & 0 \\ -1 & 1 & -1 & 2 & 0 \\ 2 & 3 & 3 & -4 & 0 \end{array}\right) \xrightarrow[\substack{III-I \\ IV+2I}]{II+I} \left(\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & 2 & 4 & 5 & 0 \\ 0 & -2 & -4 & 0 & 0 \\ 0 & 9 & 9 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[II \leftrightarrow IV]{1/9\, IV} \left(\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 \\ 0 & 2 & 4 & 5 & 0 \end{array}\right) \xrightarrow[IV-2II]{III+2II} \left(\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 5 & 0 \end{array}\right)$$

$$\xrightarrow{IV+III} \left(\begin{array}{cccc|c} -1 & 3 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \end{array}\right)$$

$\lambda=3$ NON è autovalore

$\lambda=-2$:

$$\left(\begin{array}{cccc|c} 4 & 3 & 3 & 2 & 0 \\ 1 & 4 & 1 & 3 & 0 \\ -1 & 1 & 4 & 2 & 0 \\ 2 & 3 & 3 & 1 & 0 \end{array}\right) \xrightarrow[\substack{II-4I \\ III+I \\ IV-2I}]{I \leftrightarrow II} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & -13 & -1 & -10 & 0 \\ 0 & 5 & 5 & 5 & 0 \\ 0 & -5 & 1 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[II \leftrightarrow III]{1/5\, III} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -13 & -1 & -10 & 0 \\ 0 & -5 & 1 & -5 & 0 \end{array}\right) \xrightarrow[IV+5II]{III+13II} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 12 & 3 & 0 \\ 0 & 0 & 6 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{III \leftrightarrow IV} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 12 & 3 & 0 \end{array}\right) \xrightarrow{IV - 2III} \left(\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right)$$

$\lambda = -2$ NO è autoval ∎

ESERCIZIO 2

$\mathbb{R}^4$ $\quad U: \begin{cases} 2x_1 - 3x_2 + x_3 + x_4 = 0 \\ x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 - 3x_2 + 2x_3 - 4x_4 = 0 \end{cases}$ $\quad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -7 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

a) Si determinino dimensioni, basi ed eq. cartesiane di U, W

$$U: \begin{cases} 2x_1 - 3x_2 + x_3 + x_4 = 0 \\ x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_1 - 3x_2 + 2x_3 - 4x_4 = 0 \end{cases} \qquad \left(\begin{array}{cccc|c} 2 & -3 & 1 & 1 & 0 \\ 1 & -2 & 1 & -1 & 0 \\ 1 & -3 & 2 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{I \leftrightarrow II} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 1 & 1 & 0 \\ 1 & -3 & 2 & -4 & 0 \end{array}\right) \xrightarrow[III - I]{II - 2I} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{III + II} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} a - 2b + c - d = 0 \\ b - c + 3d = 0 \end{cases} \quad *$$

$$\begin{cases} a - 2c + 6d + c - d = 0 \\ b = c - 3d \end{cases} \qquad \begin{cases} a = c - 5d \\ b = c - 3d \end{cases}$$

$$U = \left\{ \begin{pmatrix} c - 5d \\ c - 3d \\ c \\ d \end{pmatrix} \middle| \, c, d \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \dim U = 2$$

$$W: \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & 0 & 4 & -1 \\ 0 & 3 & -7 & 1 \\ 7 & 4 & 0 & -1 \end{pmatrix} \xrightarrow[IV - 7I]{II - 3I} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 7 & -1 \\ 0 & 3 & -7 & 1 \\ 0 & -3 & 7 & -1 \end{pmatrix}$$

$$\xrightarrow[IV - II]{III + II} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 7 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 7 \\ -1 \end{pmatrix} \right\rangle \qquad \dim W = 2$$

Eq cart U $\quad \begin{cases} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \end{cases}$ $\quad *$

$$\xrightarrow{III + \frac{(x_3 - x_1)}{3} II} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -3 & 7 & -1 \\ 0 & 0 & x_3 + 7/3 x_2 - 4/3 x_1 & x_4 - x_2/3 + x_1/3 \end{pmatrix}$$

$$\begin{cases} -4x_1 + 7x_2 + 3x_3 = 0 \\ x_1 - x_2 + 3x_4 = 0 \end{cases}$$

b) Si dimostri che U, W sono complementari.

dim U = 2 dim W = 2 se dim(U ∩ W) = 0

allora sono complementari

$$\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ x_1 - x_2 + 3x_4 = 0 \\ -4x_1 + 7x_2 + 3x_3 = 0 \end{cases} \qquad \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 1 & -1 & 0 & 3 & 0 \\ -4 & 7 & 3 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[IV + 4I]{III - I} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 4 & 0 \\ 0 & -1 & 7 & -4 & 0 \end{array}\right) \xrightarrow[IV + II]{III - II} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{III \leftrightarrow IV} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow U, W \text{ complementari.}$$

c) Sia θ la simmetria di $\mathbb{R}^4$ di direzione $\langle e_3, e_4 \rangle$ e asse $\langle e_2 + e_3, e_1 \rangle$. Si calcoli l'immagine del vettore $(1, 1, 1, 2)$

$\mathbb{R}^4 = \langle e_2 + e_3, e_1 \rangle + \langle e_3, e_4 \rangle$

$e_1 = 0(e_2+e_3) + 1(e_1) + 0(e_3) + 0(e_4)$

$e_2 = 1(e_2+e_3) + 0(e_1) - 1(e_3) + 0(e_4)$

$e_3 = 0(e_2+e_3) + 0(e_1) + 1(e_3) + 0(e_4)$

$e_4 = 0(e_2+e_3) + 0(e_1) + 0(e_3) + 1(e_4)$

$\vartheta(e_1) = 0(e_2+e_3) + 1(e_1) - 0(e_3) - 0(e_4) = e_1$

$\vartheta(e_2) = 1(e_2+e_3) + 0(e_1) + 1(e_3) - 0(e_4) = e_2 + 2e_3$

$\vartheta(e_3) = 0(e_2+e_3) + 0(e_1) - 1(e_3) - 0(e_4) = -e_3$

$\vartheta(e_4) = 0(e_2+e_3) + 0(e_1) - 0(e_3) - 1(e_4) = -e_4$

$$\vartheta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \vartheta\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -2 \end{pmatrix}$$

d) Si determinino le immagini di U e W.
Sono ancora complementari?

$\vartheta(U) = \langle \vartheta(u_1), \vartheta(u_2) \rangle \qquad \vartheta(W) = \langle \vartheta(w_1), \vartheta(w_2) \rangle$

$$\vartheta(u_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = u_1$$

$$\vartheta(u_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ -6 \\ -1 \end{pmatrix}$$

$$\vartheta(w_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

$$\vartheta(w_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -13 \\ 1 \end{pmatrix}$$

Sono ancora complementari se $\vartheta(u_1), \vartheta(u_2), \vartheta(w_1), \vartheta(w_2)$ generano $\mathbb{R}^4$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -5 & -3 & -6 & -1 \\ 1 & 1 & 3 & 0 \\ 0 & -3 & -13 & 1 \end{pmatrix} \xrightarrow[\text{III}-\text{I}]{\text{II}+5\text{I}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & -3 & -13 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{IV}+3/2\,\text{II}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -23/2 & 5/2 \end{pmatrix} \xrightarrow{\text{IV}+23/4\,\text{III}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5/2 \end{pmatrix}$$

Sono ancora complementari

⚠ In generale ϑ è iniettiva, potevamo già dire da mandare v. lin. indip. in v. lin. indip.

4) ESERCIZIO 3 (in E^3)

$$r: \begin{cases} x-2y+z=4 \\ x-3y=4 \end{cases} \qquad s: \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

a) Determinare una rappr. param. di r e le eq. cartesiane di s

$$r: \begin{cases} x-2y+z=4 \\ x-3y=4 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 1 & -3 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{\text{II}-\text{I}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -1 & -1 & 0 \end{array}\right) \qquad \begin{cases} a-2b+c=4 \\ b+c=0 \end{cases} \qquad \begin{cases} a=4-3c \\ b=-c \end{cases}$$

$$r: \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\rangle = \begin{cases} x=4-3\lambda \\ y=-\lambda \\ z=\lambda \end{cases}$$

$$s: \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\rangle = \begin{cases} x=1+3\mu \\ y=\mu \\ z=2-\mu \end{cases} \qquad \begin{cases} x-3y=1 \\ z+y=2 \end{cases}$$

b) Posizione reciproca di r e s (giustificando le risposte)

$r \cap s$: $\begin{cases} x-2y+z=4 \\ x-3y=4 \\ x-3y=1 \\ z+y=2 \end{cases}$ $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 1 & -3 & 0 & 4 \\ 1 & -3 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$

$\xrightarrow{IV \leftrightarrow II} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & -3 & 0 & 1 \\ 1 & -3 & 0 & 4 \end{array}\right) \xrightarrow[III-I]{IV-III} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 3 \end{array}\right)$

$\xrightarrow{III+II} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right) \xrightarrow{IV+3III} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$rk\,A=2 \qquad rk\,(A|b)=3$

sono parallele ma senza punti in comune

⚠ hanno le stesse giaciture!

Bastava verificare se $P_s \in r$ o $P_r \in s$

c) Determinare la distanza $d(r,s)$ e una coppia di punti di minima distanza

Scegliamo un punto di s e facciamo $d(P_s, r)$

1) $P_s = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\quad r: \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \left\langle \underbrace{\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}}_{=v} \right\rangle$ $\quad P_s - P_r = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} = u$

OPPURE

$d(P_s, r) = \dfrac{\| u \times v \|}{\| v \|} = \dfrac{\sqrt{22}}{\sqrt{11}} = \sqrt{2}$

$u \times v = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$

Costruisco $\tilde{\pi} \perp r$ e $P_s \in \tilde{\pi}$

$\tilde{\pi}: \begin{cases} -3x-y+z=d \\ \cdot\, P_s \in \tilde{\pi} \end{cases} \Rightarrow -3-0+2=d \Rightarrow d=-1$

$\tilde{\pi}: -3x-y+z=-1$

$H = \tilde{\pi} \cap r$

$\pi \cap r$ $\begin{cases} -3x - y - z = -1 \\ x - 2y + z = 4 \\ x - 3y = 4 \end{cases}$ $\left(\begin{array}{ccc|c} 1 & -3 & 0 & 4 \\ 1 & -2 & 1 & 4 \\ -3 & -1 & 1 & -1 \end{array}\right)$

$\xrightarrow[\text{III}+3\text{I}]{\text{II}-\text{I}} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & -10 & 1 & 11 \end{array}\right) \xrightarrow{\text{III}+10\text{II}} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 11 & 11 \end{array}\right)$

$\begin{cases} x - 3y = 4 \\ y + z = 0 \\ 11z = 11 \end{cases}$ $\begin{cases} x = 4 - 3 = 1 \\ y = -1 \\ z = 1 \end{cases}$ $H = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

d) Quali piani del fascio di centro r, intersecano s?

In quanti punti?

Un solo piano, quello che le contiene entrambe

✱ $r = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\rangle$ $P_r = \begin{pmatrix} 4 - 3\lambda \\ -\lambda \\ \lambda \end{pmatrix}$

$s = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \right\rangle$ $P_s = \begin{pmatrix} 1 + 3\mu \\ \mu \\ 2 - \mu \end{pmatrix}$

$\begin{cases} P_r - P_s \perp r \\ P_r - P_s \perp s \end{cases}$ $P_r - P_s = \begin{pmatrix} 3 - 3\lambda - 3\mu \\ -\lambda - \mu \\ -2 + \lambda + \mu \end{pmatrix}$

$\begin{cases} -9 + 9\lambda + 9\mu + \lambda + \mu - 2 + \lambda + \mu = 0 \\ \text{UGUALE} \end{cases}$ $\begin{cases} -11 + 11\lambda + 11\mu = 0 \\ \cancel{\ } \end{cases}$ $\begin{cases} \lambda = -\mu + 1 \end{cases}$

$P_r = \begin{pmatrix} 4 + 3\mu - 3 \\ +\mu - 1 \\ -\mu + 1 \end{pmatrix}$ $P_s = \begin{pmatrix} 1 + 3\mu \\ \mu \\ 2 - \mu \end{pmatrix}$

UNA COPPIA $\mu = 0$ $P_r = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $P_s = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$