PROPRIETÀ DELLA DERIVAZIONE.

$f, g : D \longrightarrow \mathbb{R}$ $\quad p$ interno a D.
$|\vec{u}| = 1$; esistano $\partial_{\vec{u}} f(p)$, $\partial_{\vec{u}} g(p)$.

- $\partial_{\vec{u}} (f+g)(p) = \partial_{\vec{u}} f(p) + \partial_{\vec{u}} g(p)$
- $\partial_{\vec{u}} (fg)(p) = \partial_{\vec{u}} f(p)\, g(p) + f(p)\, \partial_{\vec{u}} g(p)$.
- Sia $\varphi : \underset{\underset{f(D)}{\cup\!\shortmid}}{I} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, φ derivabile in $f(p)$.

Allora $\partial_{\vec{u}} (\varphi \circ f)(p) = \varphi'(f(p))\, \partial_{\vec{u}} f(p)$.

$$\begin{tikzcd} f(p) \in I \arrow[r, "\varphi"] & \varphi(f(p)) \in \mathbb{R} \\ p \in D \arrow[u, "f"] \arrow[ur, "\varphi \circ f"'] & \end{tikzcd}$$

ESEMPIO IMPORTANTE: LA NORMA.

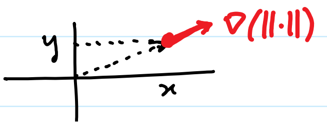
$(x,y) \longmapsto \|(x,y)\| = \sqrt{x^2+y^2}$ $\quad \begin{cases} \varphi(t) = \sqrt{t} \\ f(x,y) = x^2+y^2 \end{cases}$

Regola $\Rightarrow \|(\cdot,\cdot)\|$ derivabile rispetto ad un vettore fissato $\vec{u}$ in ogni punto (x,y) dove $x^2+y^2 \neq (0,0)$ cioè per $(x,y) \neq (0,0)$.

$$\partial_x \sqrt{x^2+y^2} = \frac{1}{2\sqrt{x^2+y^2}}\, \partial_x (x^2+y^2) = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

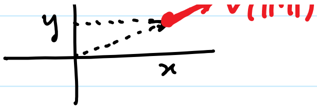
$$\partial_y \sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\nabla(\|(x,y)\|) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{1}{\|(x,y)\|}(x,y)$$



NORMA = 1

$\nabla \|x\| = \frac{x}{\|x\|}$ $\quad (x = (x_1, .., x_n))$



$$\nabla \|x\| = \frac{x}{\|x\|} \qquad (x = (x_1, \dots, x_n))$$

ES. $(x,y) \longmapsto \|(x,y)\|$ ammette derivate parziali in $(0,0)$?

? Esiste $\partial_x f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$?

$$= \lim_{h\to 0} \frac{\sqrt{h^2} - 0}{h} = \lim_{h\to 0} \frac{|h|}{h} \quad \text{NON ESISTE}$$

Analogamente $\partial_y \|\cdot\|(0,0)$ non esiste.

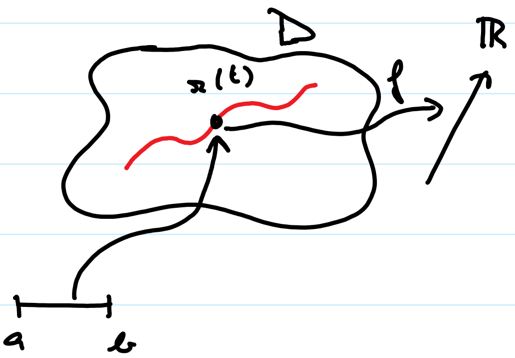
LA REGOLA DELLA CATENA

ES. $r(t) = (\cos t, t^3) \qquad f(x,y) = x^2 + y^2$

$$(f \circ r)(t) = f(r(t)) = f(\cos t, t^3) = (\cos t)^2 + (t^3)^2$$

$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ di classe $\mathcal{C}^1$ $(\partial x_i f \in \mathcal{C}(D))$

D aperto



$r : [a,b] \longrightarrow D$ curva $\mathcal{C}^1$

$r(t) = (r_1(t), \dots, r_n(t))$

$$\frac{d}{dt}(f \circ r)(t) = \underbrace{\partial_{x_1} f(r(t))\, r_1'(t)} + \dots + \underbrace{\partial_{x_n} f(r(t))\, r_n'(t)}$$

$$= \nabla f(r(t)) \bullet r'(t)$$

prodotto scalare

NB. $\frac{d}{dt}(f \circ r)(t)$: derivata di $t \mapsto f(r(t))$

$\partial_{x_1} f(r(t))$: $\partial_{x_1} f$ calcolata in $r_1(t)$

① $\partial_{x_i} f(x_1, \dots, x_n)$

② Valutarla in $r(t)$

③ Moltiplicare ciascuna per $r_i'(t)$.

ESEMPIO. $f \in C^1(\mathbb{R}^2)$, deriviamo $f(\underbrace{t^2, e^t}_{r(t)})$

$$\frac{d}{dt} f(t^2, e^t) = \partial_x f(t^2, e^t)(t^2)' + \partial_y f(t^2, e^t)\,(e^t)'$$

$$= \underbrace{\partial_x f}_{\text{der risp a } x}\underbrace{(t^2, e^t)}_{\text{calcolata in } (t^2, e^t)}\, 2t + \partial_y f(t^2, e^t)\, e^t$$

ESERCIZIO. $f(x,y) = \sin(x^2 + y)$. $\frac{d}{dt} f(e^t, \lg t) \quad (t > 0)$

Modo I: $f(e^t, \lg t) = \sin((e^t)^2 + \lg t) = \sin(e^{2t} + \lg t)$

$\frac{d}{dt} f(e^t, \lg t)$ (:= derivata $t \longmapsto f(e^t, \lg t)$)

$$= \cos(e^{2t} + \lg t)\left(2e^{2t} + \frac{1}{t}\right)$$

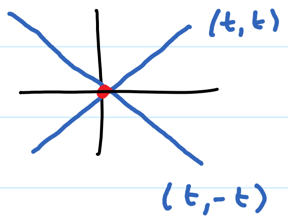
Modo II: $\partial_x f(e^t, \lg t)\,(e^t)' + \partial_y f(e^t, \lg t)\,(\lg t)'$

$$= \cos((e^t)^2 + \lg t)(2e^t)e^t + \cos((e^t)^2 + \lg t)\underbrace{(\lg t)'}_{= \frac{1}{t}}$$

ESERCIZIO. $f: \mathbb{R}^2 \to \mathbb{R}$ di classe C^1

$f(t,t) = t \qquad \forall t \in \mathbb{R}$

$f(t,-t) = 5t + t^2 \qquad \forall t \in \mathbb{R}$.



Determinare $\nabla f(0,0)$

$$\frac{d}{dt} f(t,t)\Big|_{t=0} = \frac{d}{dt} t = 1$$

$$\partial_x f(0,0)(t)'_{t=0} + \partial_y f(0,0)(t)'_{t=0} = \partial_x f(0,0) + \partial_y f(0,0)$$

$$\Rightarrow \boxed{\partial_x f(0,0) + \partial_y f(0,0) = 1}$$

$$f(t,-t) = 5t + t^2 \Rightarrow \frac{d}{dt} \underbrace{f(t,-t)}_{t=0} = (5+2t)_{t=0} = 5$$

Al tempo stesso abbiamo

$$\frac{d}{dt} f(t,-t) = \partial_x f(t,-t)\,(t)' + \partial_y f(t,-t)\,(-t)'$$

$$\frac{d}{dt} f(t,-t)_{t=0} = \partial_x f(0,0)\, 1 + \partial_y f(0,0)\,(-1)$$

$$\Rightarrow \boxed{\partial_x f(0,0) - \partial_y f(0,0) = 5}$$

Posto $a = \partial_x f(0,0)$, $b = \partial_y f(0,0)$: $\begin{cases} a+b=1 \\ a-b=5 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-2 \end{cases}$

$$\Rightarrow \boxed{\nabla f(0,0) = (3,-2)}$$

ESEMPIO. $f \in \mathcal{C}^1(D)$ $\|\vec{u}\| = 1$, $p \in D$

Allora $\boxed{\partial_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}}$ $\Rightarrow \vec{u} \longmapsto \partial_{\vec{u}} f(p)$ è lineare (del tipo $a_1 u_1 + \dots + a_n u_n$)

$$\partial_{\vec{u}} f(p) = g'(0) \qquad g(t) = f(\underbrace{p + t\vec{u}}_{r(t)})$$

$$g'(t) = \nabla f(p+t\vec{u}) \cdot \frac{d}{dt}(p+t\vec{u}) = \nabla f(p+t\vec{u}) \cdot \vec{u}.$$

ES. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$\partial_x f(0,0) = \partial_y f(0,0) = 0$$

$$\partial_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim \frac{t^3 u_1^2 u_2}{t^3} = \boxed{u_1^2 u_2}$$ NON è lineare in $\vec{u}$.
!!

$$\partial_{\vec{u}} f(0,0) = \lim_{t \to 0} \frac{f(tu_1, tu_2)}{t} = \lim \frac{\quad}{t^3} = \frac{u_1^2 u_2}{\quad}$$ è lineare in $\vec{u}$.

$\Downarrow$

f non è $\mathcal{C}^1$

$\lim_{(x,y)\to(0,0)} \partial_x f(x,y)$ NON esiste (Lez. 6)

Interpretazione del gradiente

lunedì 10 ottobre 2016 22:24

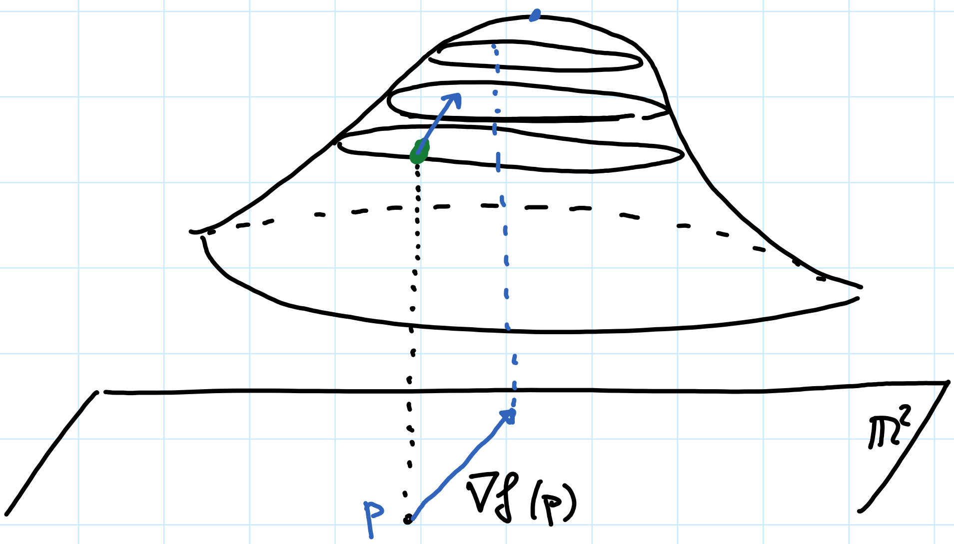
$$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u} \quad (f \in \mathcal{C}^1)$$

La direzione e il verso di MASSIMA PENDENZA.

Sia $f: D \to \mathbb{R}$ $\mathcal{C}^1$ attorno a $p \in \mathring{D}$ e $\nabla f(p) \neq 0$.

$\mathrm{MAX}\{ D_{\vec{u}} f(p) : \|\vec{u}\| = 1 \}$ raggiunto per $\vec{u} = \dfrac{\nabla f(p)}{\|\nabla f(p)\|}$;

$\mathrm{MIN}\{ D_{\vec{u}} f(p) : \|\vec{u}\| = 1 \}$ raggiunto per $\vec{u} = -\dfrac{\nabla f(p)}{\|\nabla f(p)\|}$.



Dim. Ricordiamo: $|\vec{f} \cdot \vec{u}| \leq \|\vec{f}\| \cdot \|\vec{u}\| \quad \forall \vec{f}, \underset{\neq 0}{\vec{u}} \in \mathbb{R}^n$

e l'uguale vale se $\vec{f}, \vec{u}$ proporzionali.

Si ha più precisamente $\vec{f} \cdot \vec{u} = \|\vec{f}\| \cdot \|\vec{u}\|$ se $\vec{u} = \lambda \vec{f}$, $\lambda \geq 0$;

$\vec{f} \cdot \vec{u} = -\|\vec{f}\| \cdot \|\vec{u}\|$ se $\vec{u} = -\lambda \vec{f}$, $\lambda \geq 0$

$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$ MASSIMO se $\vec{u} = \lambda \nabla f(p)$ $\lambda \geq 0$

$\|\vec{u}\| = 1 \Rightarrow \lambda = \dfrac{1}{\ldots}$

$$\|\vec{u}\| = 1 \Rightarrow \lambda = \frac{1}{\|\nabla f(p)\|}$$

$D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$ è MINIMO se $\vec{u} = -\lambda \nabla f(p)$

$$\Rightarrow \lambda = \frac{1}{\|\nabla f(p)\|}$$

$\Rightarrow D_{\vec{u}} f(p)$ MAX se $\vec{u} = \dfrac{\nabla f(p)}{\|\nabla f(p)\|}$, min. se $\vec{u} = -\dfrac{\nabla f(p)}{\|\nabla f(p)\|}$. #

ES. $f(x,y) = x^2 + y^2$ $\qquad f(3,2) = 3^2 + 2^2$

Quale direzione dobbiamo scegliere in $(3,2)$ per "salire" il più velocemente possibile?

$\dfrac{\nabla f(3,2)}{\|\nabla f(3,2)\|} = \vec{u}$ $\qquad \nabla f(x,y) = (2x, 2y)$

$\qquad \|\nabla f(x,y)\| = 2\sqrt{x^2+y^2}$

$\vec{u} = \dfrac{(2x, 2y)}{2\sqrt{x^2+y^2}} = \dfrac{(x,y)}{\sqrt{x^2+y^2}}$ nel punto $x=3$, $y=2$

$$\Rightarrow \boxed{\vec{u} = \frac{(3,2)}{\sqrt{9+4}}}$$