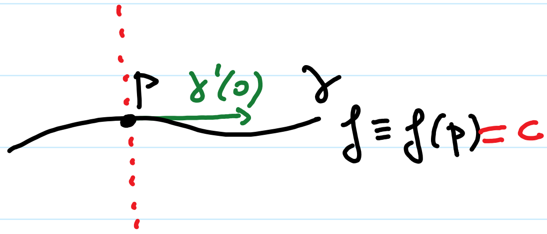
INSIEMI DI LIVELLO E GRADIENTE di una funzione.

$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ l'insieme di livello c di f è $\{x \in D : f(x) = c\}$ $\boxed{f(p) = c}$

Supponiamo che f sia $\mathscr{C}^1$ e attorno a $p \in D$ l'insieme di livello $f(p)$ sia il sostegno di una curva $\gamma :]-\delta, \delta[\longrightarrow D$

$\gamma(0) = p$



Allora $\boxed{\nabla f(p) \perp \gamma'(0)}$

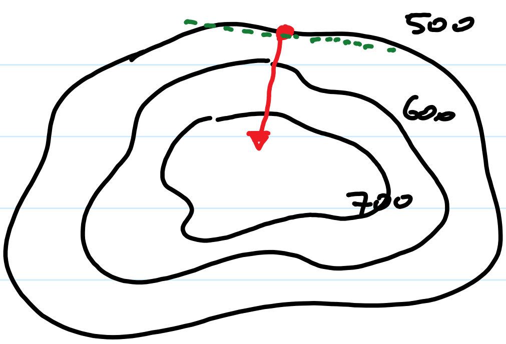
Dim. $\boxed{f(\gamma(t)) = c}$ $\forall t \in]-\delta, \delta[$ $\quad \gamma(0) = p.$

$t \longmapsto f(\gamma(t)) \equiv c$

$\Rightarrow \frac{d}{dt}(f(\gamma(t))) = 0 \Rightarrow \nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \forall t$ (↑ scalare)

Per $t = 0$: $\nabla f(\underbrace{\gamma(0)}_{p}) \cdot \gamma'(0) = 0 \Rightarrow \gamma'(0) \perp \nabla f(p)$ #

ES



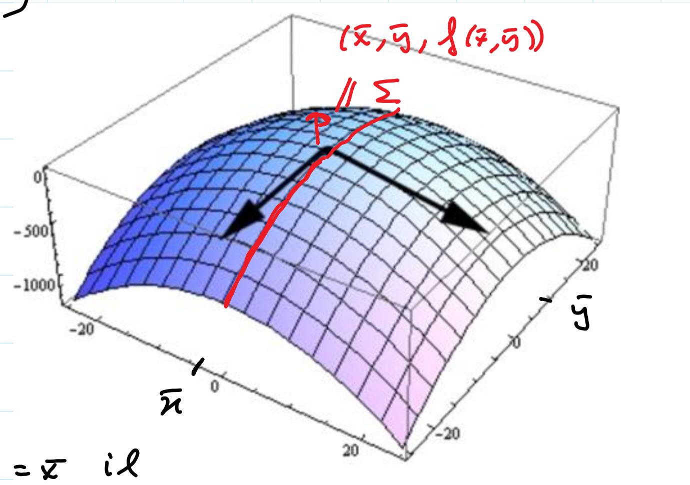
$\Sigma = \{(\bar{x}, y, f(\bar{x},y)) : y \in \mathbb{R}\}$

è il sostegno di

$y \mapsto (\bar{x}, y, f(\bar{x},y))$

Vettore tangente:

$$\boxed{(0, 1, \partial_y f(\bar{x},\bar{y}))}$$



Muovendo solo la x

otteniamo la curva

$(x, \bar{y}, f(x,\bar{y}))$; in $x = \bar{x}$ il

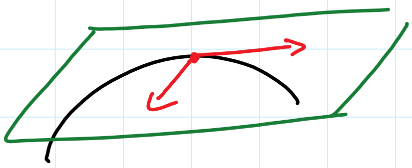
vettore tangente alla curva è $\boxed{(1, 0, \partial_x f(\bar{x},\bar{y}))}$

Piano passante per $P = (\bar{x}, \bar{y}, f(\bar{x},\bar{y}))$ generato dai

vettori $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f(\bar{x},\bar{y}) \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ \partial_y f(\bar{x},\bar{y}) \end{pmatrix}$: ($\bar{z}$)

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \det\left[\begin{pmatrix} x-\bar{x} \\ y-\bar{y} \\ z-\bar{z} \end{pmatrix}, \vec{u}, \vec{v} \right] = 0 \right\}$$

$$\begin{vmatrix} x-\bar{x} & 1 & 0 \\ y-\bar{y} & 0 & 1 \\ z-f(\bar{x},\bar{y}) & \partial_x f(\bar{x},\bar{y}) & \partial_y f(\bar{x},\bar{y}) \end{vmatrix} = 0$$



$$(x-\bar{x}) \begin{vmatrix} 0 & 1 \\ \partial_x f & 0 \end{vmatrix} - (y-\bar{y}) \begin{vmatrix} 1 & 0 \\ \partial_x f & \partial_y f \end{vmatrix} + (z - f(\bar{x},\bar{y})) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$-\partial_x f(\bar{x},\bar{y})(x-\bar{x}) - \partial_y f(\bar{x},\bar{y})(y-\bar{y}) + z - f(\bar{x},\bar{y}) = 0$$

$$\boxed{z = f(\bar{x},\bar{y}) + \partial_x f(\bar{x},\bar{y})(x-\bar{x}) + \partial_y f(\bar{x},\bar{y})(y-\bar{y})}$$

DEF. Se f è di classe $\boxed{C^1(D)}$ e $p \in D$ aperto il piano tangente al grafico di f in p è il piano passante per $P = (p, f(p))$ generato dai vettori $(1, 0, \partial_{x_1} f(p))$; $(0, 1, \partial_{x_2} f(p))$

Si tratta dell'insieme dei punti (x_1, x_2, z):

Si tratta dell'insieme dei punti (x_1, x_2, z):

$$z - f(p) = \partial_{x_1} f(p)(x_1 - p_1) + \partial_{x_2} f(p)(x_2 - p_2)$$
$$z = f(p) + \nabla f(p) \cdot (x - p) \qquad x = (x_1, x_2)$$

ES. $f(x_1, x_2) = x_1^4 + x_2^2 \qquad x_1 = 1, \quad x_2 = 2$

$f(1,2) = 1^4 + 2^2 = \boxed{5}$ Piano tangente al grafico di f nel punto $(1, 2, f(1,2)) = (1,2,5)$:

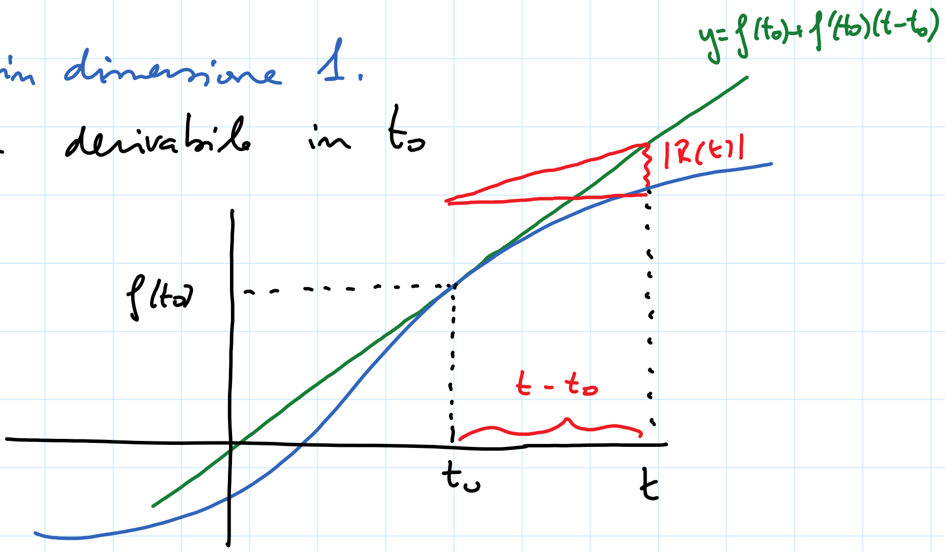
$$\left\{(x_1, x_2, z) : z = f(1,2) + \nabla f(1,2) \cdot \begin{pmatrix} x_1 - 1 \\ x_2 - 2 \end{pmatrix}\right\}$$
$$: z = f(1,2) + \partial_{x_1} f(1,2)(x_1 - 1) + \partial_{x_2} f(1,2)(x_2 - 2)$$

$\partial_{x_1} f(x_1, x_2) = 4x_1^3$
$\partial_{x_2} f(x_1, x_2) = 2x_2$

$$\boxed{z = 5 + 4(x_1 - 1) + 4(x_2 - 2)}$$

Derivata in dimensione 1.

$f: \mathbb{R} \to \mathbb{R}$ derivabile in t_0



$$\frac{f(t) - f(t_0)}{t - t_0} \xrightarrow[t \to t_0]{} f'(t_0) \quad \text{cioè} \quad \frac{f(t) - f(t_0)}{t - t_0} = f'(t_0) + \varepsilon(t)$$
con $\varepsilon(t) \xrightarrow[t \to t_0]{} 0$

$$f(t) - f(t_0) = f'(t_0)(t - t_0) + \underbrace{\varepsilon(t)(t - t_0)}_{R(t)}$$

$$f(t) = f(t_0) + f'(t_0)(t - t_0) + R(t) \quad \text{con} \quad \frac{R(t)}{t - t_0} = \varepsilon(t) \xrightarrow[t \to t_0]{} 0$$

DEF. Sia $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$, p interno a D. Esista $\nabla f(p)$. Si dice che f è differenziabile in p se

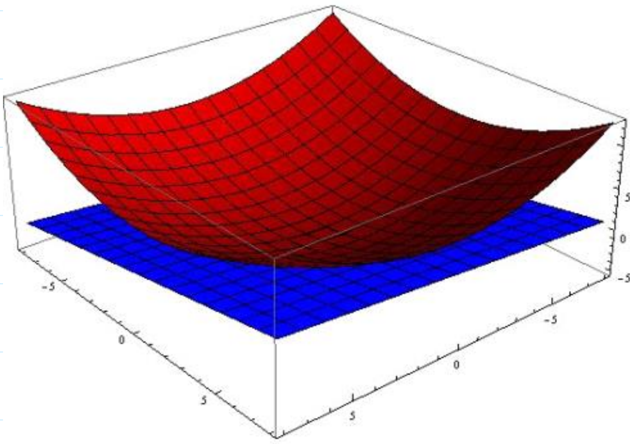
$$f(x) = \underbrace{f(p) + \nabla f(p) \cdot (x - p)} + R(x) \quad \text{con} \quad \frac{R(x)}{\|x - p\|} \xrightarrow[x \to p]{} 0$$

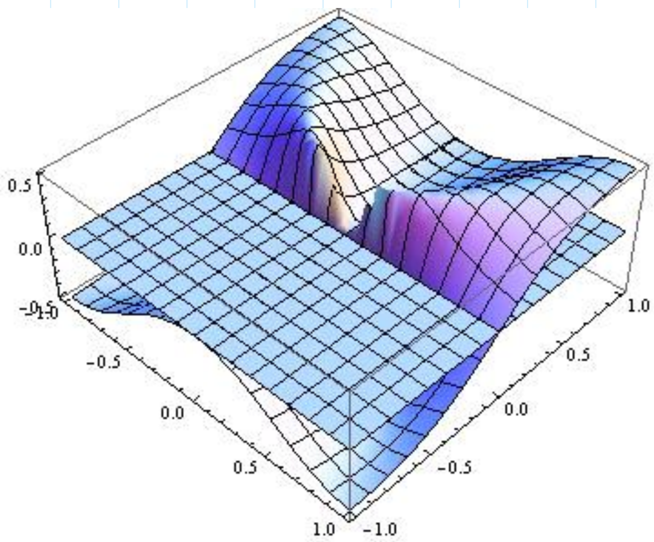
$z = f(p) + \nabla f(p)\cdot(x-p)$ è il piano tangente.

Significato: f differenziabile in $p \Leftrightarrow$ la funzione

$$x \longmapsto f(p) + \nabla f(p)\cdot(x-p)$$

approssima bene f attorno a p.





$\exists\, \partial_{x_i} f$ continue

TEOREMA. $f \in \mathcal{C}^1(D)$, D aperto $\Rightarrow$ f differenziabile in D

ES. $f(x,y) = x^2 + y^2$ è differenziabile in ogni punto

ES. f differenziabile in $(1,2)$ $\quad f(1,2) = 4$;
$\partial_x f(1,2) = 8$, $\partial_y f(1,2) = -3$.
Approssimiamo (usando il piano tangente al grafico di f in $(1,2,f(1,2))$) il valore $f(1.2, 1.8)$

$$f(1.2,\,1.8) \approx \underbrace{f(1,2)}_{4} + \underbrace{\partial_x f(1,2)}_{8}\,(1.2-1) + \underbrace{\partial_y f(1,2)}_{-3}\,(1.8-2)$$

PROP. f differenziabile in $p \Rightarrow f$ continua in p.

Infatti $f(x) = f(p) + \nabla f(p)\cdot \underbrace{(x-p)}_{\to 0} + R(x)$

$$R(x) = \underbrace{\frac{R(x)}{\|x-p\|}}_{\to 0}\ \underbrace{\|x-p\|}_{\to 0}$$

$$\Rightarrow \lim_{x\to p} f(x) = f(p) \quad \#$$

ESEMPIO. $f(x,y) = \begin{cases} xy/x^2+y^2 & (x,y)\neq(0,0) \\ 0 \text{ se } (x,y)=(0,0) \end{cases}$ Non è continua in $(0,0)$

$\Rightarrow$ non è differenziabile in $(0,0)$.

PROP. f differenziabile in $p \Rightarrow$

$\forall \vec{u},\ \|\vec{u}\|=1,\quad D_{\vec{u}} f(p) = \nabla f(p)\cdot \vec{u}$

Dim. ? $\displaystyle\lim_{t\to 0} \frac{f(p+t\vec{u}) - f(p)}{t}$.

f diff in $p \Rightarrow f(x) = f(p) + \nabla f(p)\cdot(x-p) + R(x)$, $\displaystyle\lim_{x\to p}\frac{R(x)}{\|x-p\|} = 0$.

$x = p + t\vec{u}$:

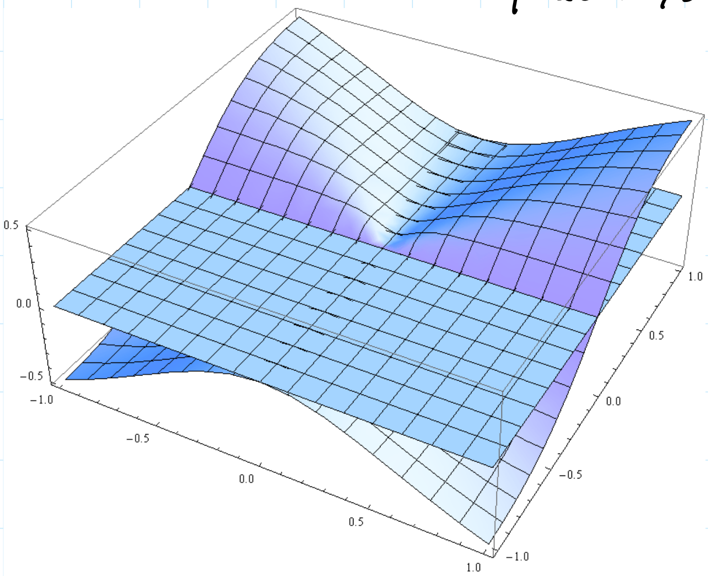
$$f(p+t\vec{u}) - f(p) = \nabla f(p)\cdot t\vec{u} + R(p+t\vec{u}) \qquad \lim_{t\to 0}\frac{R(p+t\vec{u})}{|t|} = 0$$

$$\frac{f(p+t\vec{u})-f(p)}{t} = \underbrace{\nabla f(p)\cdot\vec{u}}_{\to \nabla f(p)\cdot\vec{u}} + \underbrace{\frac{R(p+t\vec{u})}{t}}_{\to 0} \longrightarrow \nabla f(p)\cdot\vec{u}, \quad \#$$

ES. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & \text{altrimenti} \end{cases}$

- f è continua in $(0,0)$? Sì: $\left|\dfrac{x^2y}{x^2+y^2}\right| = |x|\left|\dfrac{xy}{x^2+y^2}\right| \le \dfrac{|x|}{2} \xrightarrow[x\to 0]{} 0$

- f è continua in $(0,0)$? Sì: $\left|\frac{x^2y}{x^2+y^2}\right| = |x|\left|\frac{xy}{x^2+y^2}\right| \le \frac{|x|}{2} \xrightarrow[x\to 0]{} 0$
- $\partial_{\vec{u}} f(0,0) = ?\ \lim_{t\to 0} \frac{f(t\vec{u}) - f(0)}{t} = \lim_{t\to 0} \frac{t^3 u_1^2 u_2}{t^3} = u_1^2 u_2$
- f è differenziabile in $(0,0)$? NO perché f differenziabile $\Rightarrow \partial_{\vec{u}} f(0,0) = a u_1 + b u_2$ per qualche $a, b \in \mathbb{R}$

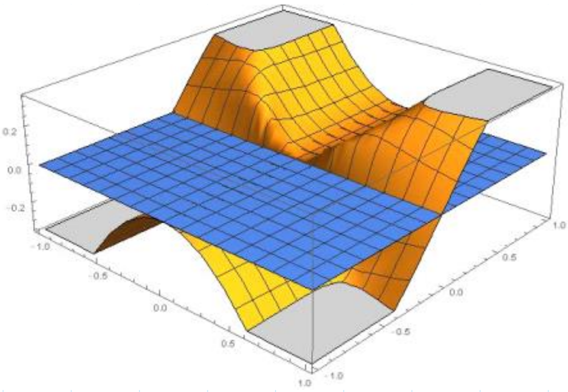


RIASSUNTO:

$$f \text{ è } \mathcal{C}^1$$
$$\Downarrow$$
$$f \text{ differenziabile}$$
$$\swarrow \qquad \searrow$$
f continua $\qquad\qquad \partial_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}$

ESERCIZIO. $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2}\sqrt{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & \text{in } (0,0). \end{cases}$

- f continua in $(0,0)$
- $\partial_{\vec{u}} f(0,0)$ lineari in $\vec{u}$
- f non è differenziabile in $(0,0)$



- Continuità: $\left|\frac{x^2 y}{x^4+y^2}\right| \le \frac{1}{2}$ $\quad \left(\left|\frac{XY}{X^2+Y^2}\right| \le \frac{1}{2}\right)$

$$\Rightarrow \quad |f(x,y)| \le \frac{1}{2}\sqrt{x^2+y^2} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$\Rightarrow f$ è continua in $(0,0)$.

- $$D_{\vec{u}} f(0,0) \stackrel{?}{=} \lim_{t\to 0} \frac{f(t\vec{u}) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3 u_1^2 u_2}{t^4 u_1^4 + t^2 u_2^2}\sqrt{t^2u_1^2+t^2u_2^2}}{t}$$

$\vec{u} = (u_1, u_2)$

$$= \lim_{t\to 0} \frac{t\, u_1^2 u_2}{t^2 u_1^4 + u_2^2} \frac{|t|}{t}$$

Se $u_2 = 0$: $D_{\vec{u}} f(0,0) = 0$

Se $u_2 \ne 0$ $\quad \left|\frac{t u_1^2 u_2}{t^2 u_1^4 + u_2^2}\right| \to 0 \Rightarrow D_{\vec{u}} f(0,0) = 0$.

$\boxed{D_{\vec{u}} f(0,0) = 0}$ $\quad \forall \vec{u}$.

- f differenziabile in $(0,0) \Leftrightarrow$

$$f(x,y) = \underset{0}{f(0,0)} + \underset{0}{\nabla f(0,0)} \cdot \begin{pmatrix} x-0 \\ y-0 \end{pmatrix} + R(x,y)$$

e $\frac{R(x,y)}{\|(x,y)\|} \xrightarrow[(x,y)\to(0,0)]{} 0$

$f(x,y) = R(x,y)$ e $\frac{R(x,y)}{\sqrt{x^2+y^2}} \longrightarrow (0,0)$?

f differenziabile $\Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$.

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{x^2 y}{x^4+y^2}$$ NON ha limite in (0,0)

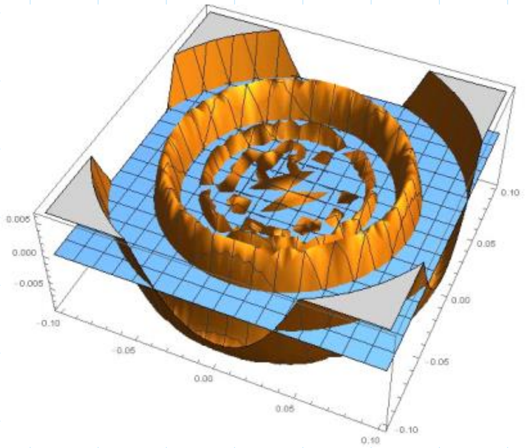
$(x,y) \neq (0,0)$

$\Downarrow$

f Non è differenziabile in (0,0).

ES. $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 \quad se & (x,y) = (0,0) \end{cases}$

f è diff in (0,0) MA NON $\mathcal{C}^1$.



ES. $f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 \quad \text{se} & (x,y)=(0,0) \end{cases}$

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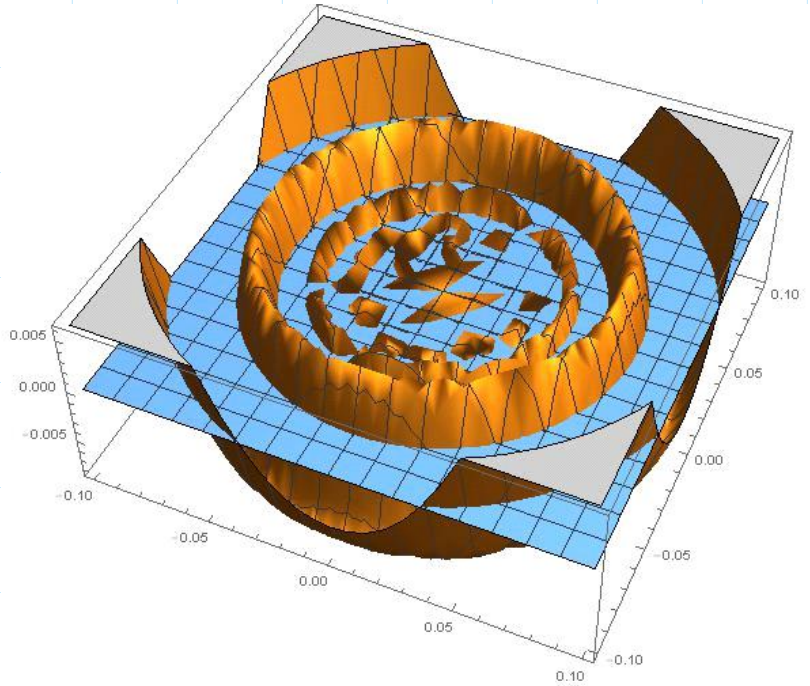
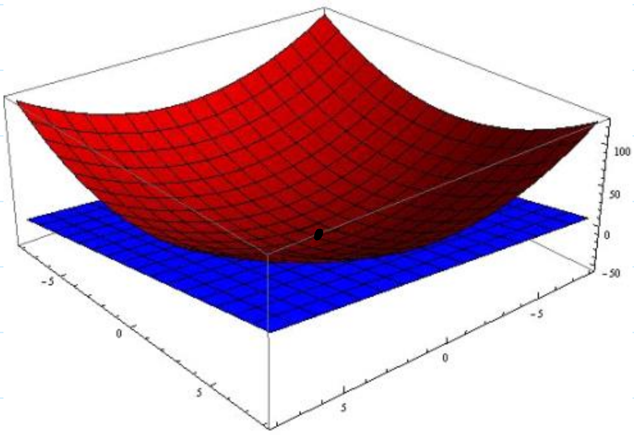


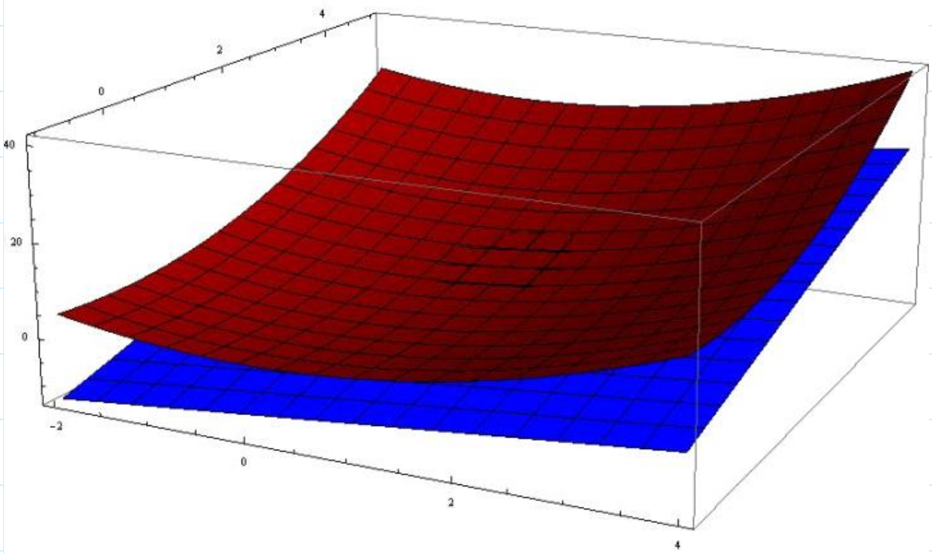
Disegni sul piano tangente

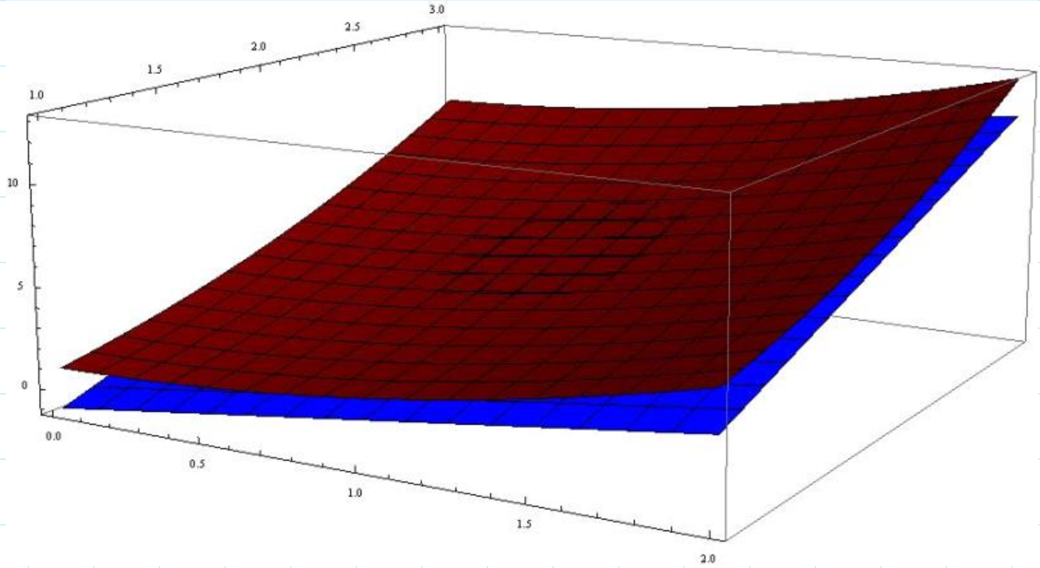
martedì 12 ottobre 2010 19:34

Piano tangente a z=x^2+y^2 in (1,2) con fattore di scala sempre più dettagliato

$$f(x,y) = x^2 + y^2$$

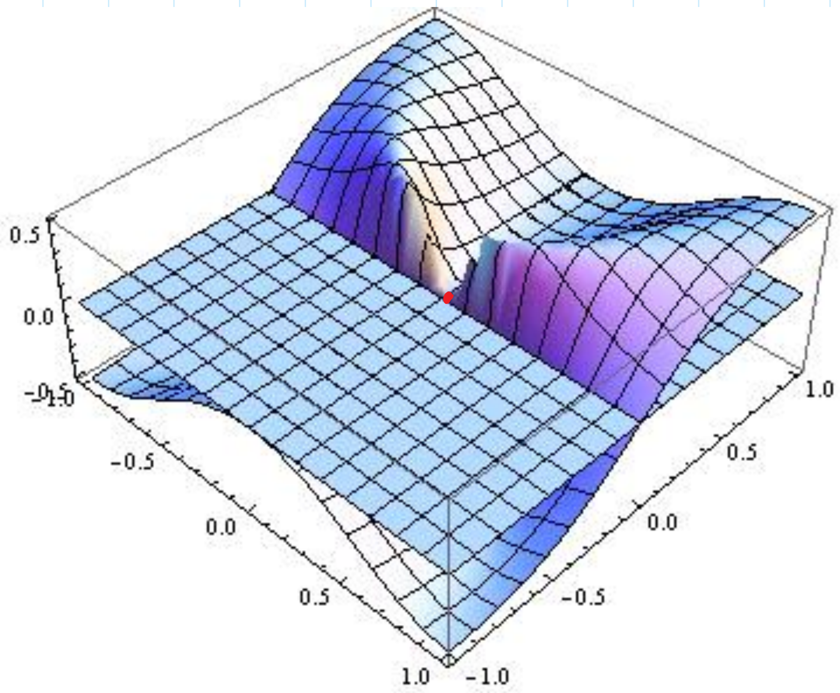






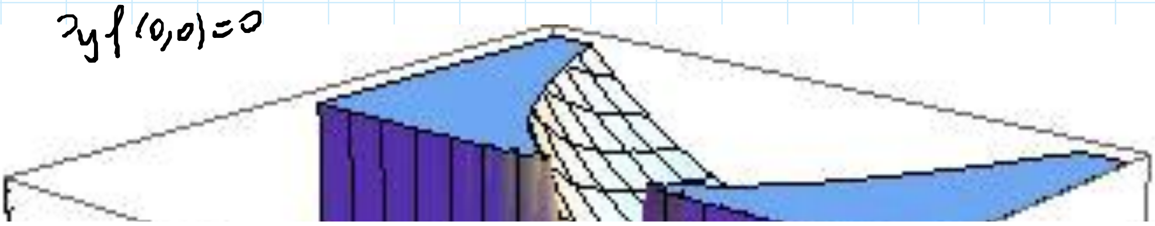
Esempio di funzione non differenziabile con derivate parziali nulle in (0,0)

x^2y/(x^4+y^2)

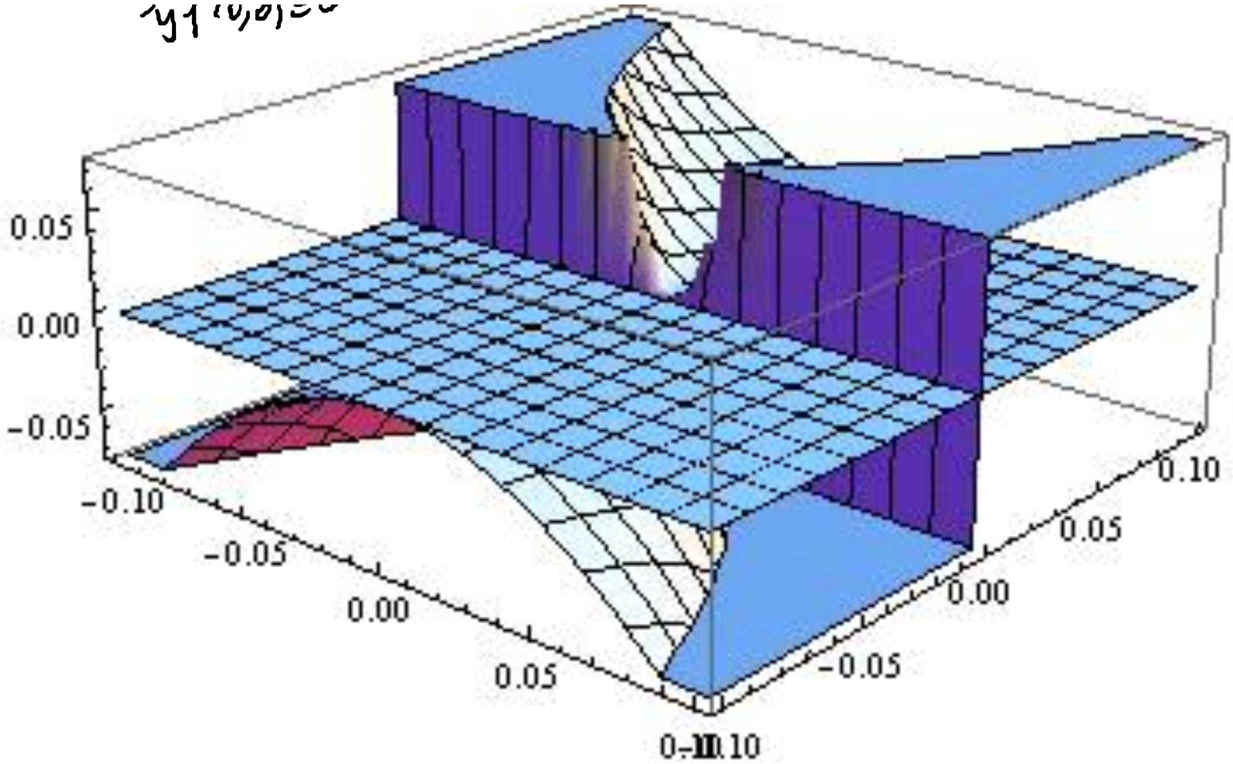


$\partial_x f(0,0)=0$ $\quad z=0$ (candidato a essere il piano tangente).

$\partial_y f(0,0)=0$

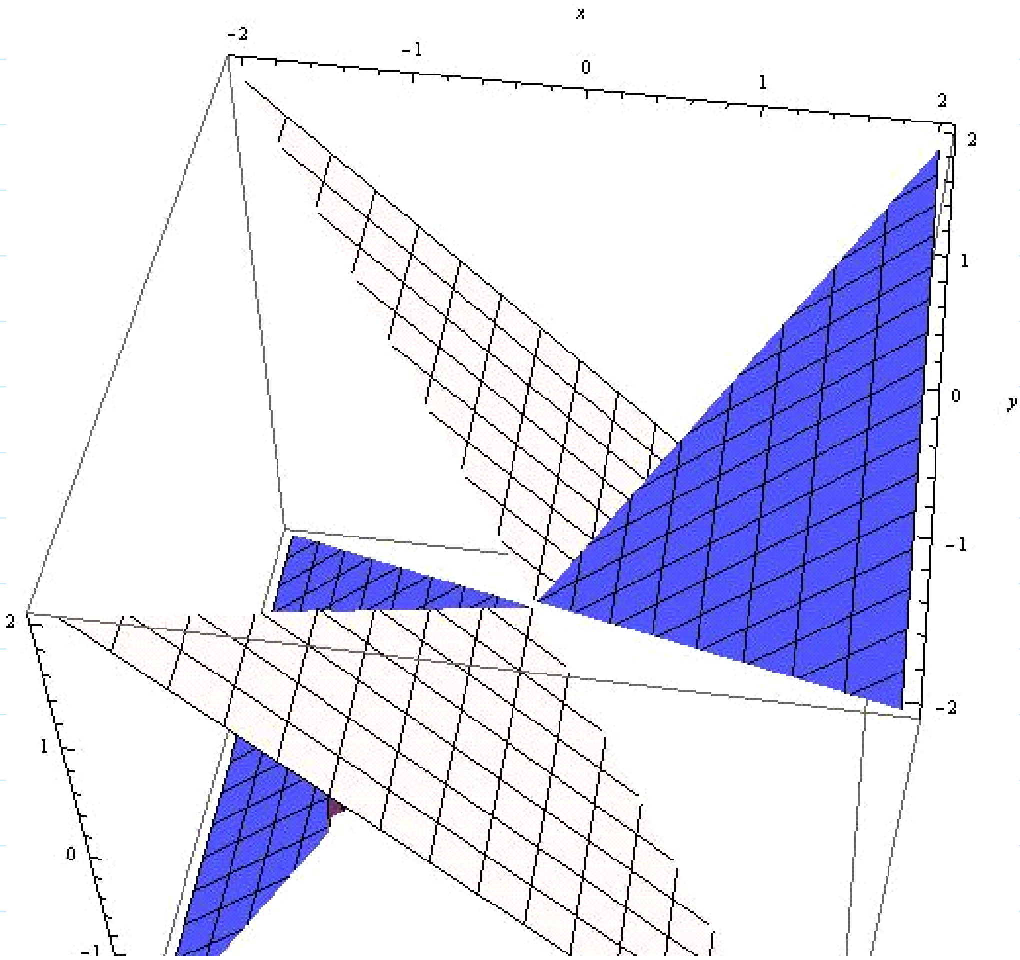


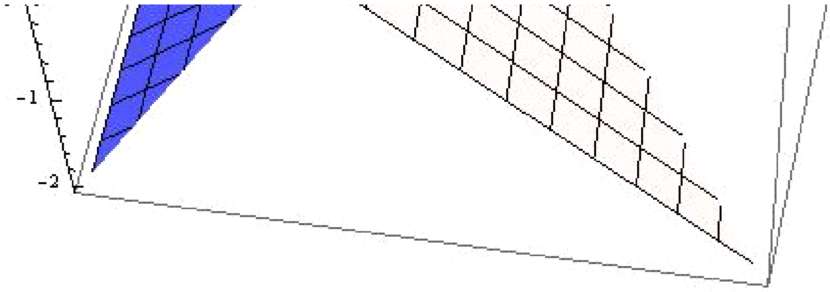
$y(0,0)=0$



x Boole[Abs[y] < Abs[x]] - x Boole[Abs[x] <= Abs[y]]}

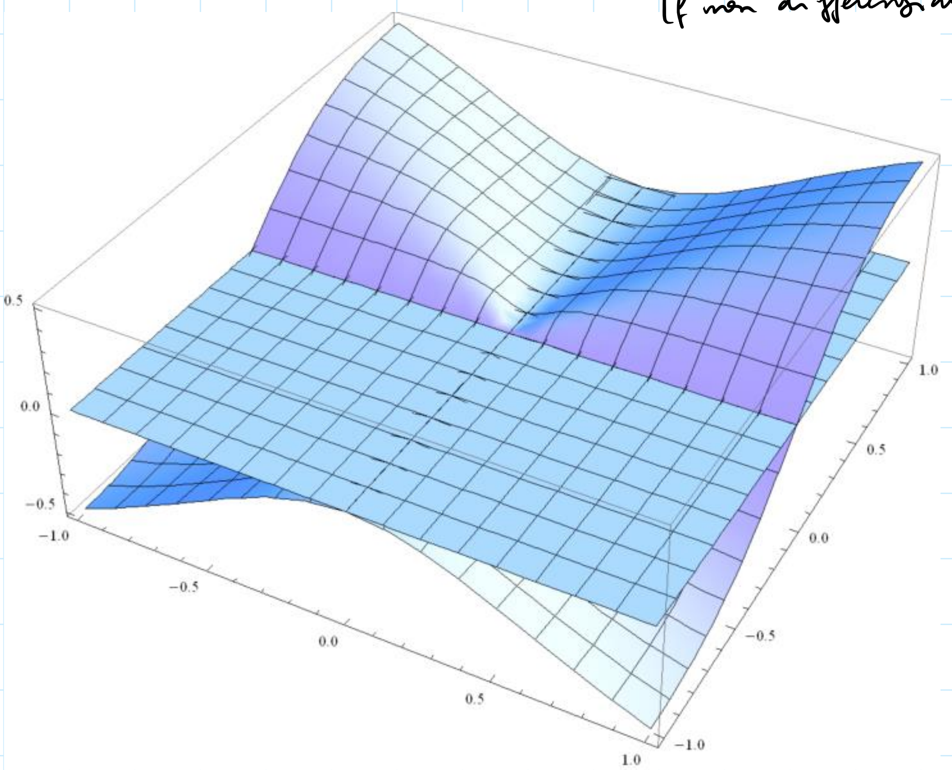
$$f(x,y) = \begin{cases} x & \text{se } |y| < |x| \\ -x & \text{se } |x| \le |y| \end{cases}$$





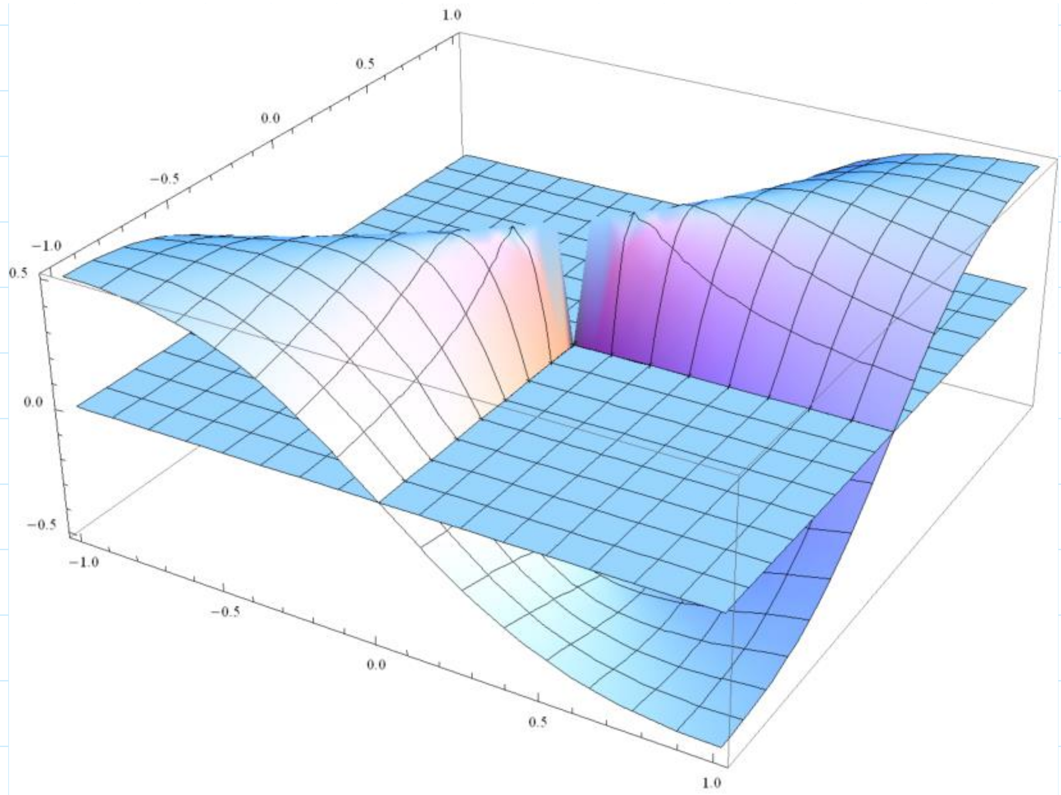
$$f(x,y) = \begin{cases} x^2 y/(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & \text{altr.} \end{cases}$$

$$\begin{cases} f \text{ continua in } 0 \\ f \text{ ha derivate direzionali} \\ f \text{ non differenziabile} \end{cases}$$



$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{altrimenti} \end{cases}$$

$$\begin{cases} f \text{ non continua in } (0,0) \\ \partial_x f(0,0) = \partial_y f(0,0) = 0 \\ f \text{ non differenziabile in } (0,0) \end{cases}$$



1.0
0.5
0.0
−0.5
−1.0
0.5
0.0
−0.5
−1.0
−0.5
0.0
0.5
1.0

ESERCIZIO

Siano $\varphi : I \subseteq \mathbb{R} \longrightarrow J \subseteq \mathbb{R}$, $f : D \subseteq \mathbb{R}^n \longrightarrow I$
I, D aperti, φ e f di classe $\mathcal{C}^1$.
Allora $\nabla(\varphi \circ f)(x) = \varphi'(f(x)) \nabla f(x)$.

Sol. $\partial_{x_i} (\varphi \circ f)(x) = \varphi'(f(x)) \partial_{x_i} f(x) \quad \forall i$

$$\Rightarrow \quad \nabla(\varphi \circ f)(x) = \varphi'(f(x)) \nabla f(x)$$

ESERCIZIO. $\varphi : [0, +\infty[\longrightarrow \mathbb{R}$ $\mathcal{C}^1$.
$\forall x \neq 0$ in $\mathbb{R}^n$, $\nabla(\varphi(\|x\|)) = ?$

Si ha
$$\boxed{\begin{aligned} \nabla(\varphi(\|x\|)) &= \varphi'(\|x\|) \nabla \|x\| \\ &= \varphi'(\|x\|) \frac{x}{\|x\|}. \end{aligned}}$$

ES. Trovare $\varphi : [0, +\infty[\longrightarrow \mathbb{R}$ affinché

$$\nabla(\varphi(\|x\|)) \quad \frac{K}{\ldots} \quad \frac{x}{\ldots} .$$

$$\nabla(\varphi(\|x\|)) = \frac{K}{\|x\|^2} \frac{x}{\|x\|}.$$

Per l'esercizio precedente si ha

$\nabla\varphi(\|x\|) = \varphi'(\|x\|)\frac{x}{\|x\|}$: basta scegliere

φ in modo tale che $\varphi'(r) = \frac{K}{r^2}$ cioè

$$\varphi(r) = -\frac{K}{r}.$$