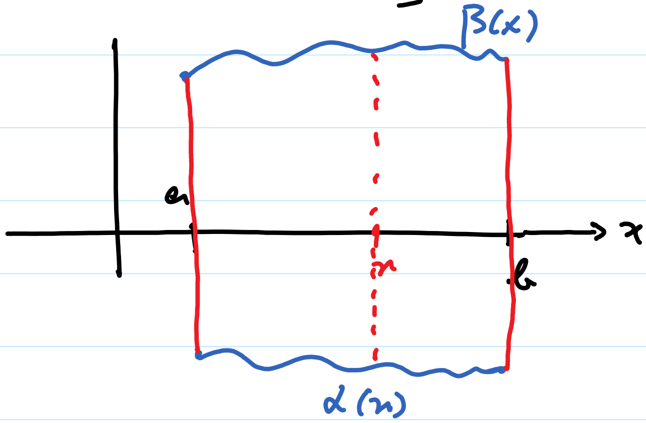
INTEGRALI SU DOMINI SEMPLICI

DEF. $D \subseteq \mathbb{R}^2$ è semplice rispetto a x

$$D = \{(x,y) : x \in [a,b] \quad \alpha(x) \le y \le \beta(x)\}$$

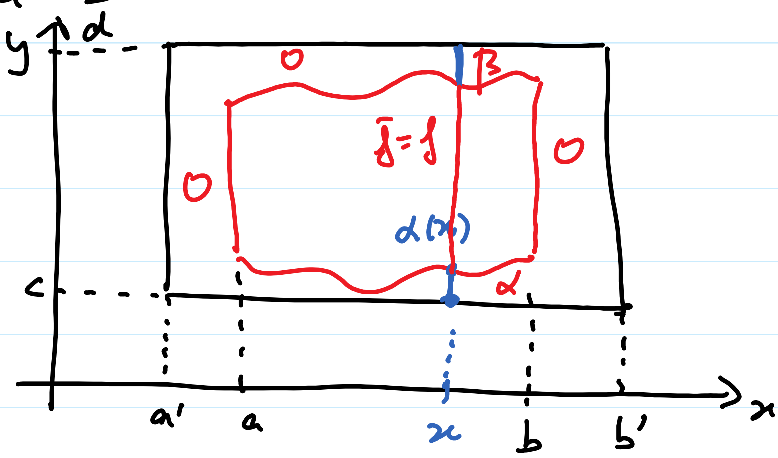


Formule di riduzione:

$$\int_D f(x,y)\,dx\,dy = \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx.$$

Dim. $\int_D f(x,y)\,dx\,dy = \int_R \bar{f}_R(x,y)\,dx\,dy$

$$\bar{f}_R(x,y) = \begin{cases} f(x,y) & \text{in } D \\ 0 & \text{fuori di } D \end{cases}$$



$$\int \bar{f}_R(x,y)\,dx\,dy = \int_{a'}^{b'} \left\{ \int_c^d \bar{f}_R(x,y)\,dy \right\} dx = \int_a^b \int_c^d \bar{f}_R(x,y)\,dy\,dx$$

x fissato in $[a,b]$: $\bar{f}_R(x,y) = 0 \quad y \in [c, \alpha(x)]$

$$\tilde{f}_R(x,y) = 0 \qquad y \in [\beta(x), d]$$

$$\int_c^d \tilde{f}_R(x,y)\,dy = \int_{\alpha(x)}^{\beta(x)} \tilde{f}_R(x,y)\,dy$$

$$= \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy$$

$$\Rightarrow \int_D f(x,y)\,dx\,dy = \int_a^b \left\{ \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right\} dx.$$

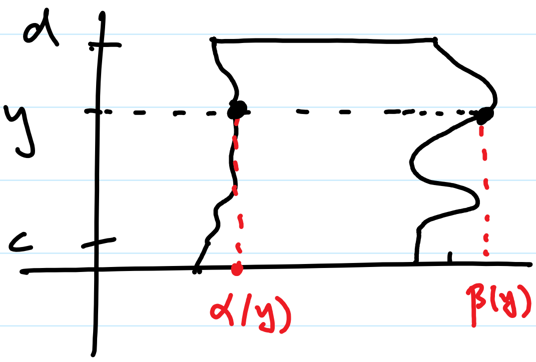
DEF. D è semplice rispetto a ~~y~~ x

$$D = \{ (x,y) : \; y \in [c,d], \quad \alpha(y) \le x \le \beta(y) \}$$

$$\int_D f(x,y)\,dx\,dy$$

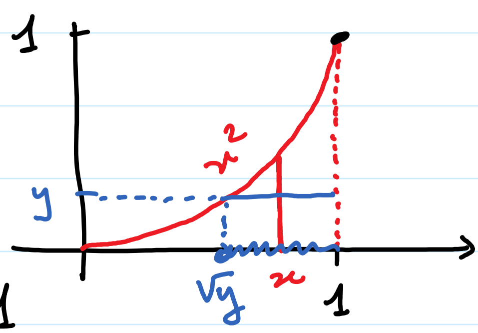
$$\parallel$$

$$\int_c^d \left\{ \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx \right\} dy$$



ES. $D = \{ (x,y) : \; 0 \le x \le 1, \quad 0 \le y \le x^2 \}$

D è semplice rispetto ad x



$$0 \le x \le 1 \Rightarrow 0 \le y \le x^2 \le 1$$

$$\Downarrow$$

$$x \ge \sqrt{y}$$

$$D = \{ (x,y) : \; y \in [0,1] \quad \sqrt{y} \le x \le 1 \}$$

Esercizi

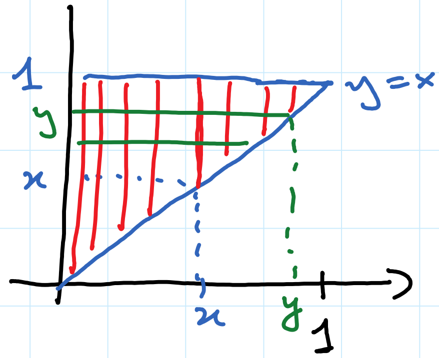
mercoledì 28 ottobre 2015 14:26

INTEGRALI SU DOMINI SEMPLICI

ESERCIZIO. $\int_D \sin y^2 \, dx\,dy$ $\quad D=\{(x,y):\ 0\le x\le 1,\ x\le y\le 1\}$

D è semplice rispetto ad x.

$$\int_D \sin y^2 \, dx\,dy = \int_0^1 \left\{\int_x^1 \sin y^2 \, dy\right\} dx$$



$0\le x\le y\le 1 \Rightarrow 0\le x\le y$ $\quad$ y fissato in $[0,1]$

$D=\{(x,y):\ y\in[0,1]:\ 0\le x\le y\}$

$\Rightarrow$ D è semplice rispetto ad y. La formula di riduzione porge

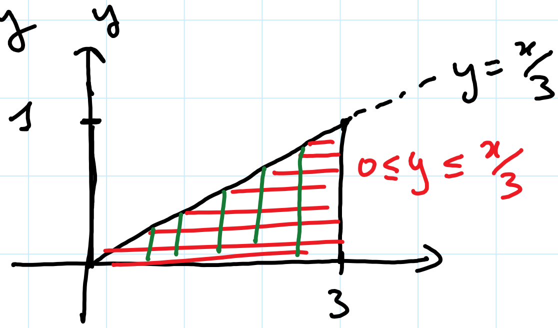
$$\int_D \sin y^2 \, dx\,dy = \int_0^1 \left\{\int_0^y \sin y^2 \, dx\right\} dy = \int_0^1 \left[x \sin y^2\right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin y^2 \, dy = \left[-\frac{\cos y^2}{2}\right]_{y=0}^{y=1} = -\frac{\cos 1}{2} + \frac{\cos 0}{2}$$

$$= \frac{1-\cos 1}{2}.$$

ESEMPIO. $D=\{(x,y):\ y\in[0,1],\ 3y\le x\le 3\}$ $\to$ è semplice rispetto a y

$f(x,y)=e^{x^2}$. Calcolare $\int_D e^{x^2} dx\,dy$



$y\in[0,1] \Rightarrow 0\le x\le 3$

$3y\le x \Rightarrow y\le x/3$

$D=\{(x,y):\ x\in[0,3]:\ 0\le y\le x/3\}$ è semplice rispetto ad x.

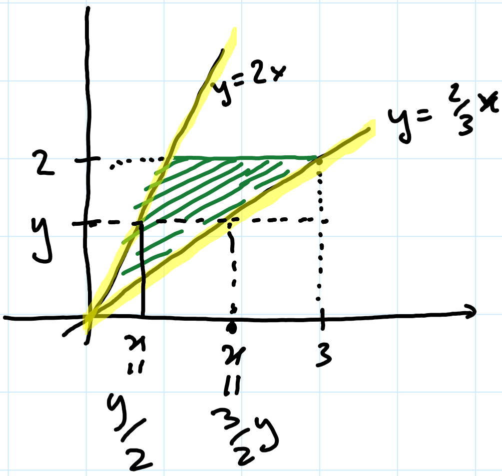
$$\int_D e^{x^2} dx\,dy = \int_0^3 \left\{\int_0^{x/3} e^{x^2} dy\right\} dx = \int_0^3 \left[y\, e^{x^2}\right]_{y=0}^{y=x/3} dx$$

$$= \int_0^3 \frac{1}{3} x e^{x^2} dx = \frac{1}{6}\left[e^{x^2}\right]_{x=0}^{x=3} = \frac{1}{6}\left(e^9-1\right).$$

$$= \int_0 \frac{1}{3} x e^{x^2} dx = \frac{1}{6}\left[e^{x^2}\right]_{x=0} = \frac{1}{6}\left(e^3 - 1\right).$$

ESEMPIO.

$$\int_D e^{x/y}\, dx\,dy \qquad D \text{ in figura.}$$



Scriviamo D come dominio semplice rispetto ad y:

$$D = \left\{ (x,y) : y \in [0,2] : \frac{y}{2} \le x \le \frac{3}{2} y \right\}$$

$$\int_D e^{x/y}\, dx\,dy = \int_0^2 \left\{ \int_{y/2}^{3/2 y} e^{x/y}\, dx \right\} dy$$

$$= \int_0^2 \left[y e^{x/y} \right]_{x=y/2}^{x=3/2 y} dy = \int_0^2 y\left(e^{3/2} - e^{1/2}\right) dy$$

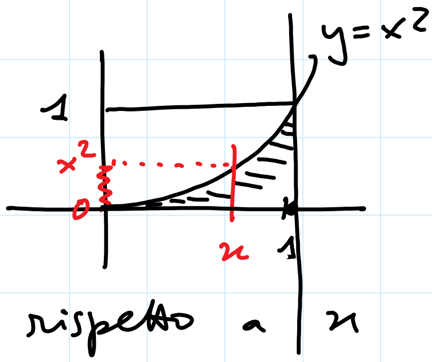
$$= \left(e^{3/2} - e^{1/2}\right) \int_0^2 y\, dy = \left(e^{3/2} - e^{1/2}\right)\left[\frac{1}{2} y^2\right]_0^2$$

$$= \left(e^{3/2} - e^{1/2}\right) \cdot 2 \qquad \#$$

ESERCIZIO. Calcolare $\int_D x^3 \cos(xy)\, dx\,dy$

$$D = \left\{ (x,y) : y \ge 0, \ \sqrt{y} \le x \le 1 \right\}$$

$$D = \left\{ (x,y) : x \ge 0, \ x \le 1, \ 0 \le y \le x^2 \right\}$$



Integrando prima risp. a y e poi rispetto a x

si ottiene

$$\int_D x^3\cos(xy)\,dx\,dy = \int_0^1 \left\{\int_0^{x^2} x^3\cos(xy)\,dy\right\} dx$$

$$\int_0^{x^2} x^3\cos(xy)\,dy = x^3\int_0^{x^2}\cos(xy)\,dy = \underset{(x\neq 0)}{x^3}\left[\frac{1}{x}\sin(xy)\right]_0^{x^2}$$

$$= \frac{x^3}{x}\left(\sin x^3 - \sin 0\right) = x^2\sin x^3$$

$$\int_0^1 x^2\sin x^3\,dx = -\frac{1}{3}\left[\cos x^3\right]_{x=0}^{x=1} = -\frac{1}{3}(\cos 1 - \cos 0)$$

$$= \frac{1}{3}(1-\cos 1)$$

ESERCIZIO. $\displaystyle\int_D \frac{\sin y^2}{y}\,dx\,dy$ $\qquad D = \{(x,y) : y\in[0,\sqrt{\pi}],\ 0\le x\le y^2\}$

F. di riduzione

$$\int_D \frac{\sin y^2}{y}\,dx\,dy = \int_0^{\sqrt{\pi}} \int_0^{y^2} \frac{\sin y^2}{y}\,dx\;dy$$

$$= \int_0^{\sqrt{\pi}} \left[x\,\frac{\sin y^2}{y}\right]_{x=0}^{x=y^2} dy$$

$$= \int_0^{\sqrt{\pi}} y\sin y^2\,dy = -\frac{1}{2}\left[\cos y^2\right]_0^{\sqrt{\pi}} = -\frac{1}{2}(\cos\pi - \cos 0)$$

$$= -\frac{1}{2}(-1-1) = 1.$$

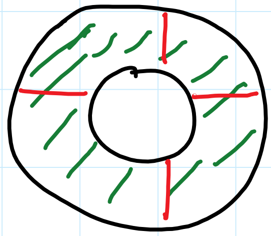
! $\displaystyle\int_a^b \Big\{\underbrace{\int_c^d f(x,y)\,dy}_{\text{funzione della variabile } x}\Big\}\,dx \in \mathbb{R}$

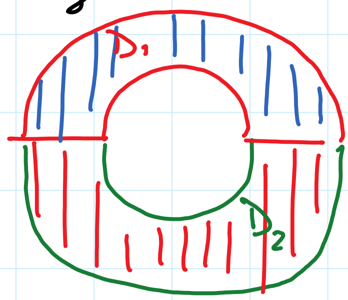
CASO di DOMINI non SEMPLICI

Se $D_1, \dots, D_n$ sono sottoinsiemi di $\mathbb{R}^2$ limitati a due a due disgiunti

$$\int_{D_1 \cup \dots \cup D_n} f(x,y)\,dx\,dy = \sum_{i=1}^{n} \int_{D_i} f(x,y)\,dx\,dy.$$

In generale si cerca di decomporre un dominio in unione di domini semplici

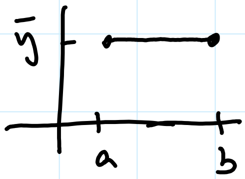




OSS: Integrale doppio su insiemi "1-dimensionali".

Sia $f: [a,b] \times [c,d] \longrightarrow \mathbb{R}$ continua.

Su ogni segmento $L = [a,b] \times \{\bar{y}\}$ si ha



$$\int_L f(x,y)\,dx\,dy = 0.$$

Infatti

$$\int_L f(x,y)\,dx\,dy = \int_a^b \underbrace{\int_{\bar{y}}^{\bar{y}} f(x,y)\,dy}_{=0}\,dx = 0.$$

CAMBIO DI VARIABILE

Ripasso (Analisi 1): $\int_a^b f(x)\,dx \qquad x=\varphi(u)$

$\varphi:[c,d]\to[a,b]$ derivabile con derivata continua

$\varphi(c)=a$, $\varphi(d)=b$ $\qquad \int_a^b f(x)\,dx=\int_c^d f(\varphi(u))\varphi'(u)\,du \qquad (1)$

$\varphi(c)=b$, $\varphi(d)=a$ $\qquad \int_a^b f(x)\,dx=\int_d^c f(\varphi(u))\varphi'(u)\,du=-\int_c^d f(\varphi(u))\varphi'(u)\,du \quad (2)$

Se φ è biiettiva $\Rightarrow$ φ è monotona

(1): φ crescente $\Rightarrow \varphi'\geq 0 \qquad \int_c^d f(\varphi(u))\varphi'(u)\,du=\boxed{\int_c^d f(\varphi(u))|\varphi'(u)|\,du}$

(2) φ decrescente

$$-\int_c^d f(\varphi(u))\varphi'(u)\,du=\int_c^d f(\varphi(u))(-\varphi'(u))\,du$$

$$=\boxed{\int_c^d f(\varphi(u))|\varphi'(u)|\,du}$$

$$\int_a^b f(x)\,dx=\int_c^d f(\varphi(u))|\varphi'(u)|\,du.$$

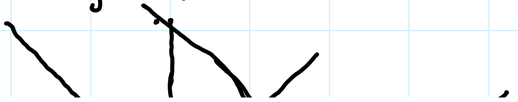
PERCHÉ CAMBIARE VARIABILI?

① Semplificare il dominio

$$\int_D f(x,y)\,dx\,dy$$

$$D=\{(x,y):\ 1\leq \underbrace{x+y}_{u}\leq 3,\ \ -1\leq \underbrace{x-y}_{v}\leq 2\}.$$

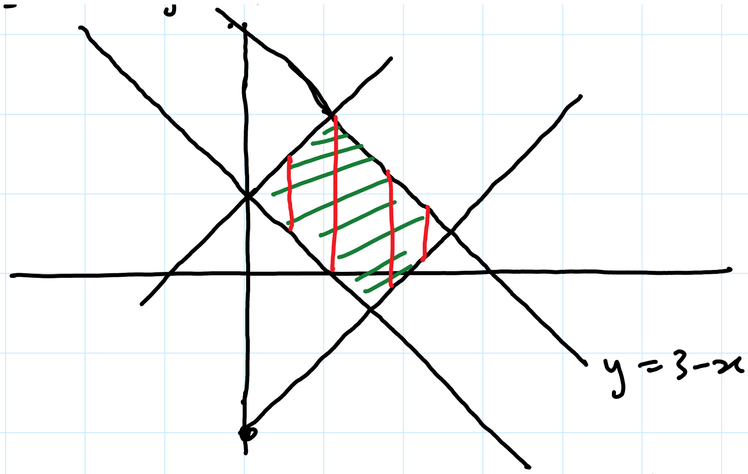
$1-x\leq y\leq 3-x \qquad\qquad x-2\leq y\leq x+1$



$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

$$2x = u+v$$

$$\begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$$



$$\boxed{(x,y) = \varphi(u,v)} = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$1 \le x+y \le 3$ e $-1 \le x-y \le 2 \iff u \in [1,3]$
$v \in [-1,2]$

② Semplificare l'integrando

ES $$\int_D (x+y)\, e^{xy}\, dx\,dy : \quad \boxed{\begin{matrix} u = x+y \\ v = xy \end{matrix} \quad ?}$$

$$\int_? u\, e^{v} \dots \, du\, dv$$

MATRICE JACOBIANA.

$$\varphi : E \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(u,v) \longmapsto (\varphi_1(u,v), \varphi_2(u,v)) \qquad \varphi_1, \varphi_2 \in \mathcal{C}^1$$

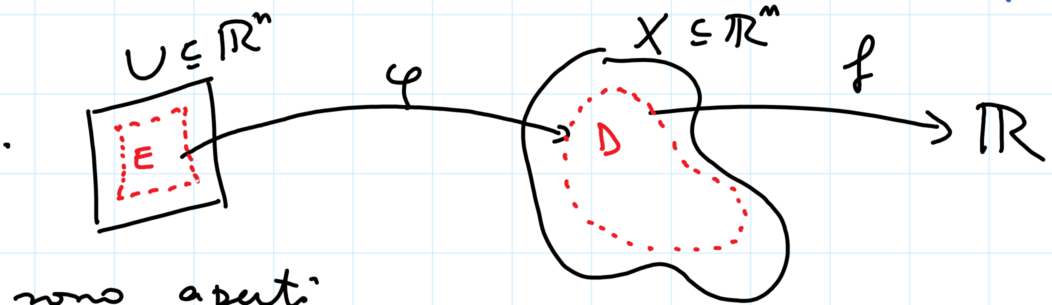
la matrice jacobiana di φ è la matrice le cui righe sono i gradienti di φ_1 e di φ_2

$$\varphi'(u,v) := \mathrm{Jac}\, \varphi(u,v) := \begin{array}{c} \begin{pmatrix} \partial_u \varphi_1 & \partial_v \varphi_1 \\ \partial_u \varphi_2 & \partial_v \varphi_2 \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow \\ \partial_u & \quad \partial_v \end{matrix} \end{array} = \begin{pmatrix} \nabla \varphi_1(u,v) \\ \hline \nabla \varphi_2(u,v) \end{pmatrix}$$

ESEMPIO. $\varphi(u,v) = (2u^2 v,\ e^u \cos v)$

$$\mathrm{Jac}\,\varphi(u,v) = \begin{pmatrix} 4uv & 2u^2 \\ e^u \cos v & -e^u \sin v \end{pmatrix}$$

TEOREMA DI CAMBIAMENTO DI VARIABILE ($n=2$).



- U, X sono aperti
- φ di classe $\mathcal{C}^1$: $\varphi = (\varphi_1, \varphi_2)$ con $\varphi_1, \varphi_2 \in \mathcal{C}^1$
- φ sia biiettiva
- $\det \varphi'(u,v) \neq 0 \quad \forall (u,v) \in U$ ($\Leftrightarrow \varphi^{-1}$ è $\mathcal{C}^1$)

$f : D \to \mathbb{R}$ integrabile.

Allora $\boxed{\forall D \subseteq X}$

$$\int_D f(x,y)\,dx\,dy = \int_E f(\varphi(u,v))\,|\det \varphi'(u,v)|\,du\,dv$$

dove $E = \varphi^{-1}(D) = \{(u,v) \in U : \varphi(u,v) \in D\}$

$= \{(u,v) \in U \ (x(u,v), y(u,v)) \in D\}$.

(cioè $\varphi(E) = D$)

OSS:

1) Si richiede che U, X siano aperti: ciò serve anche per poter parlare di derivabilità di f e φ.

2) La formula vale su qualunque sottoinsieme D di X. Così ad esempio se si pone

$\begin{cases} u = x+y \\ v = x-y \end{cases}$ cioè $\begin{cases} x = \frac{u+v}{2} = \varphi_1(u,v) \\ y = \frac{u-v}{2} = \varphi_2(u,v) \end{cases}$ la applicazione

$\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(u,v) \longmapsto \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ è definita su $\mathbb{R}^2$,

poi uno applica la formula nel dominio di interesse. Ad esempio

$$\int_{\substack{1 \le x+y \le 3 \\ -2 \le x-y \le 1}} f(x,y)\,dx\,dy = \int_{[1,3]_u \times [-2,1]_v} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) |\varphi'(u,v)|\,du\,dv$$

$\begin{pmatrix} u = x+y \\ v = x-y \end{pmatrix}$

ed è $|\varphi'(u,v)| = \left|\det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$

da cui $$\int_{\substack{1 \le x+y \le 3 \\ -2 \le x-y \le 1}} f(x,y)\,dx\,dy = \int_{[1,3] \times [-2,1]} f\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \frac{1}{2}\,du\,dv.$$

ESEMPIO (Formula delle Aree).

Per $f = 1$ la formula porge

$$\boxed{\text{Area}(D) = \int_D 1\,dx\,dy = \int_E |\varphi'(u,v)|\,du\,dv, \quad E = \varphi^{-1}(D).} \quad (*)$$

INTERPRETAZIONE

Se $f \approx \underset{c}{\text{costante}}$ in D si ha $\int_D f\,dx\,dy = c \times \text{Area}(D)$.

Cambio di variabile

Registrazione audio avviata: 22:16 sabato 22 ottobre 2016

Se $(\bar{u},\bar{v}) \in E$ è un intorno sufficientemente piccolo di $(\bar{u},\bar{v})$ si ha piccolo di $(\bar u,\bar v)$ si ha

$$\int_E |\varphi'(u,v)|\,du\,dv \approx \int_E |\varphi'(\bar u,\bar v)|\,du\,dv = |\varphi'(\bar u,\bar v)| \times \text{Area}(E).$$

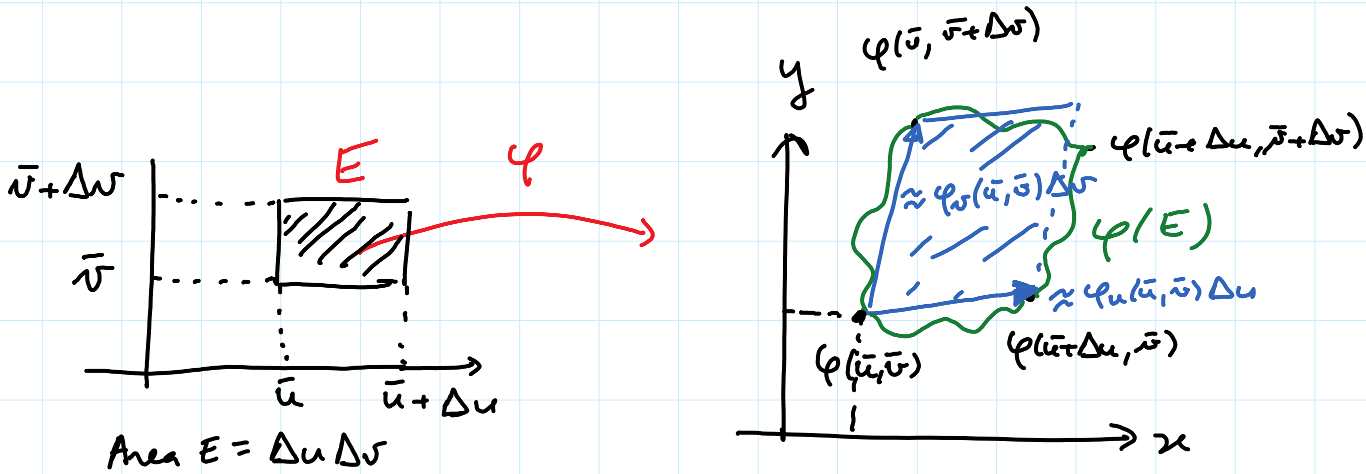
Analogamente, se $(\bar x,\bar y)=\varphi(\bar u,\bar v)$ è

$$\int_D f(x,y)\,dx\,dy \approx \int_D f(\bar x,\bar y)\,dx\,dy = f(\bar x,\bar y)\times \text{Area}(D).$$

Per $f=1$ la formula $(*)$ implica quindi

$$\text{Area}(\overset{\overset{D}{\shortparallel}}{\varphi(E)}) = |\varphi'(\bar u,\bar v)| \times \text{Area}(E):$$ tale formula mostra come si modificano le aree in piccola scala.

Eccone una interpretazione:



Area $E = \Delta u \Delta v$

Area $\varphi(E)$

$$\approx |\varphi_u(\bar u,\bar v)\Delta u \times \varphi_v(\bar u,\bar v)\Delta v|$$

$$= |\Delta u|\,|\Delta v|\,\underbrace{|\varphi_u(\bar u,\bar v)\times\varphi_v(\bar u,\bar v)|}_{\shortparallel \\ |\varphi'(\bar u,\bar v)|}$$

$$= |\varphi'(\bar u,\bar v)| \times \text{Area}(E).$$

ESEMPIO. $\varphi(u,v) = (au, bv) \qquad a, b \neq 0$

$$\varphi'(u,v) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \det \varphi'(u,v) = ab$$

$$\int_D f(x,y)\,dx\,dy = \int_E f(au, bv)\,|ab|\,du\,dv$$

$$E = \{(u,v) : (au, bv) \in D\}.$$

Caso particolare: $a = b$.

$$\int_D 1\,dx\,dy = \int_E a^2\,du\,dv$$

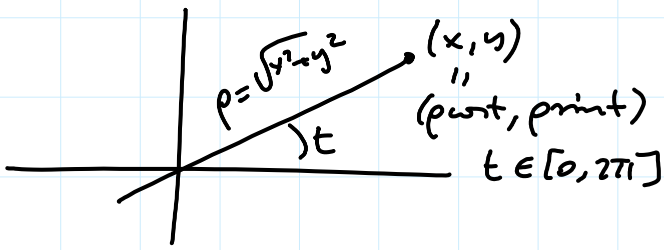
$$E = \{(u,v) : (au, av) \in D\} = \{(u,v) : a(u,v) \in D\} = \frac{1}{a} D$$

$$= \int_D 1\,dx\,dy = a^2 \int_{\frac{1}{a}D} 1\,du\,dv$$

$$\text{Area}(D) = a^2\,\text{Area}\left(\tfrac{1}{a}D\right) \Rightarrow \boxed{\text{Area}(\lambda D) = \lambda^2\,\text{Area}(D)}$$

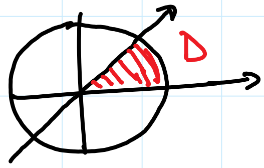
ESEMPIO (Coordinate polari).

$$\begin{cases} x = \rho\cos t \\ y = \rho\sin t \end{cases} \qquad \rho \geq 0, \quad t \in [0, 2\pi]$$



$$\boxed{\int_D f(x,y)\,dx\,dy = \int_E f(\rho\cos t, \rho\sin t)\,\rho\,d\rho\,dt \qquad E = \{(\rho,t) : \rho\ge 0,\ t\in[0,2\pi] : (\rho\cos t, \rho\sin t)\in D\}}$$

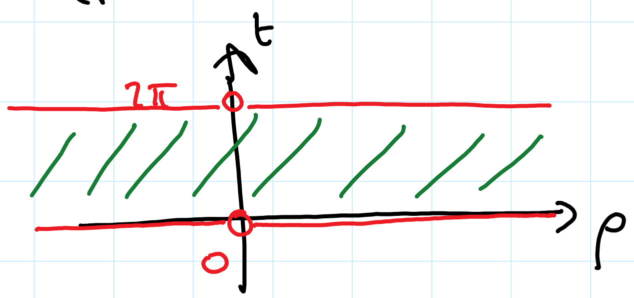
ES $D = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1,\ 0\le y\le x\}$

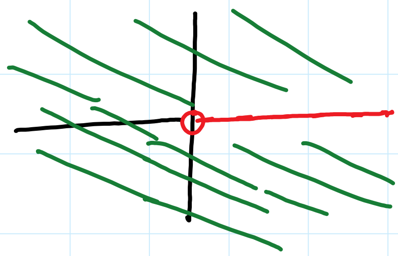


$$\begin{cases} x = \rho\cos t \\ y = \rho\sin t \end{cases} \qquad \rho\ge 0,\ t\in[0,2\pi] \iff \rho\in[0,1],\ t\in[0,\tfrac{\pi}{4}]$$
$$\Rightarrow E = [0,1]\times[0,\tfrac{\pi}{4}]$$

Dim. Consideriamo la mappa

$$U =]0,+\infty[\times]0,2\pi[\xrightarrow{\varphi} \mathbb{R}^2\setminus\{(x,0) : x\ge 0\} = X$$
$$(\rho,\ t)\longmapsto(\rho\cos t, \rho\sin t)$$



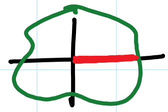


- U e X sono aperti
- φ è C^1 su U
- φ è biiettiva

- Matrice jacobiana di $(\rho,t)\xrightarrow{\varphi}(\rho\cos t, \rho\sin t)$

$$\varphi'(\rho,t) = \begin{pmatrix} \cos t & -\rho\sin t \\ \sin t & \rho\cos t \end{pmatrix}$$

$$\det \varphi'(\rho, t) = \rho\cos^2 t + \rho\sin^2 t = \rho > 0 \text{ su } U.$$



Ora sia $D \subseteq \mathbb{R}^2$. Notiamo che $D \cap \{(x,0) : x \geq 0\}$ è contenuto in un segmento. Quindi

$$\int_D f(x,y)\,dx\,dy = \int_{D \setminus \{(x,0):\, x \geq 0\}} f(x,y)\,dx\,dy$$

$$= \int_{E'} f(\rho\cos t, \rho\sin t)\,\rho\,d\rho\,dt$$

(Th. di camb. di variabile)

$$E' = \{(\rho, t) \in U : (\rho\cos t, \rho\sin t) \in D \setminus \{(x,0):\, x \geq 0\}\}$$

$$= \int_E f(\rho\cos t, \rho\sin t)\,\rho\,d\rho\,dt$$

$$E = \{(\rho, t) : \rho \geq 0,\ t \in [0, 2\pi],\ (\rho\cos t, \rho\sin t) \in D\},$$

dato che $E \setminus E'$ è contenuto nelle rette $t = 0$, $t = 2\pi$ e $\rho = 0$ (su cui l'integrale di $f(\rho\cos t, \rho\sin t)\rho$ vale 0)

Es. $\displaystyle\int_{x^2+y^2 \leq 1} e^{x^2+y^2}\,dx\,dy =$

1) Semplice risp. ad x: $x \in [-1,1]$ $y \in [-\sqrt{1-x^2}, \sqrt{1-x^2}]$

$$\int_{-1}^{1} \left\{ \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2} dy \right\} dx$$

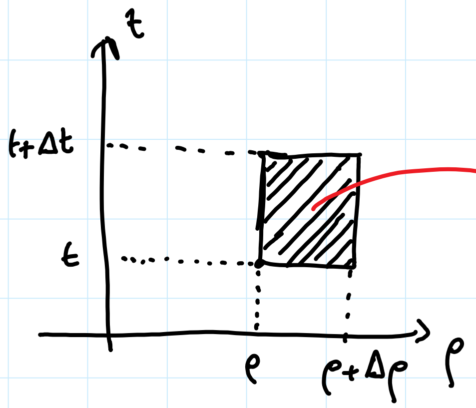
$$\parallel$$

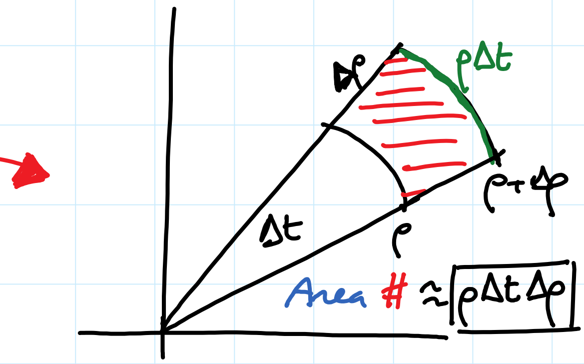
$$\int_{-1}^{1} e^{x^2} \underbrace{\int_{?}^{?} e^{y^2} dy}_{\text{non è integrabile elementarmente}} \, dx$$

2) $$\int_{x^2+y^2 \le 1} e^{x^2+y^2} dx\,dy = \int_{\substack{\rho \in [0,1] \\ t \in [0,2\pi]}} e^{\rho^2 \cos^2 t + \rho^2 \sin^2 t} \rho \, d\rho \, dt$$

$$= \int_{[0,1]\times[0,2\pi]} \rho e^{\rho^2} d\rho\, dt = \left(\int_0^1 \rho e^{\rho^2} d\rho \right) \times \left(\int_0^{2\pi} 1\, dt \right) = 2\pi \left[\frac{1}{2} e^{\rho^2} \right]_0^1$$
$$= \pi(e-1) \#$$

Interpretazione del termine $\rho\, d\rho\, dt$:





Oss. Talvolta conviene considerare coordinate polari di centro $(a,b) \neq (0,0)$. In tal caso si pone
$$\begin{cases} x = a + \rho\cos t \\ y = b + \rho\sin t \end{cases}$$