FAMP_2016_17_Lezione13

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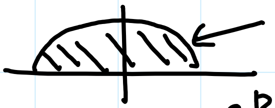
ESEMPIO. Area del disco di raggio R:

Modo 1:



Trap: $x \mapsto \sqrt{R^2-x^2}$

$$\text{Area } B[0,R] = 2\int_0^R \sqrt{R^2-x^2}\,dx.$$

Modo 2.

$$\text{Area } B[0,R] = \int_{B[0,R]} 1\,dx\,dy \overset{\substack{x=\rho\cos t\\ y=\rho\sin t}}{=} \int_{\substack{0\le\rho\le R\\ 0\le t\le 2\pi}} \rho\,d\rho\,dt$$

$$= \left(\int_0^R \rho\,d\rho\right)\left(\int_0^{2\pi} 1\,dt\right) = \frac{1}{2}R^2 \times 2\pi = \pi R^2 \quad \#$$

ESEMPIO (ellisse). $a>0$, $b>0$

$$E = \left\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1\right\}.$$

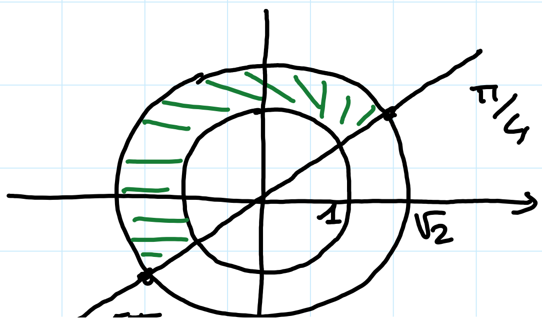
Porto $u = \dfrac{x}{a}$, $v=\dfrac{y}{b}$ $\qquad (x,y)\in E \iff u^2+v^2\le 1$

$x = au \qquad y = bv$

$$(u,v) \overset{\varphi}{\longmapsto} (au,bv) \qquad \text{Area } E = \int_E 1\,dx\,dy = \int_{B[0,1]} ab\,du\,dv = \pi ab.$$

ES. $D = (x,y):\ 1\le x^2+y^2\le 2,\ \ y\ge x\}$

$$\int_D x\,dx\,dy$$

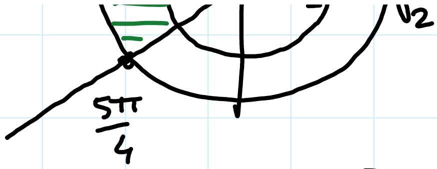


$x=\rho\cos t$, $y=\rho\sin t$

$1 \le \rho \le \sqrt{2}$ $\qquad t\in\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$x = \rho \cos t$, $y = \rho \sin t$

$1 \le \rho \le \sqrt{2}$, $t \in [\pi/4, 5\pi/4]$



$$\int_D x\,dx\,dy = \int_{[1,\sqrt{2}]_\rho \times [\pi/4, 5\pi/4]_t} \rho \cos t\, \rho \, d\rho\, dt = \left(\int_1^{\sqrt{2}} \rho^2\, d\rho\right)\left(\int_{\pi/4}^{5\pi/4} \cos t\, dt\right)$$

$$= \frac{1}{3}\left[\rho^3\right]_1^{\sqrt{2}} \left[\sin t\right]_{\pi/4}^{5\pi/4}$$

$$= \frac{1}{3}(2\sqrt{2}-1)\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= -\frac{\sqrt{2}}{3}(2\sqrt{2}-1).$$

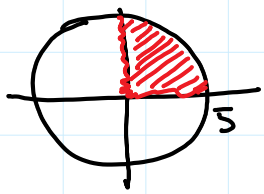
ES. Interpretare l'integrale iterato come l'integrale di una funzione su una opportuna regione, calcolarlo.

$$\alpha = \int_0^5 \left\{ \int_0^{\sqrt{25-x^2}} x\, dy \right\} dx.$$

Se $\alpha = \int_D x\,dx\,dy \Rightarrow D = \{(x,y) : x \in [0,5],\ 0 \le y \le \sqrt{25-x^2}\}$

$$D = \{(x,y) : x \in [0,5],\ 0 \le y,\ x^2+y^2 \le 25\}$$

$$\int_D x\,dx\,dy = \int_{[0,5]_\rho \times [0,\pi/2]_t} (\rho \cos t)\rho\, d\rho\, dt$$



$$= \left(\int_0^5 \rho^2\, d\rho\right) \times \int_0^{\pi/2} \cos t\, dt = \frac{1}{3}5^3 \times 1 = \frac{53}{25}$$

$$= \left(\int_0 \rho^2 \, d\rho \right) \times \int_0 \cos t \, dt = \frac{1}{3} 5^3 \times 1 = \frac{5^5}{25}$$

OSS. Un integrale iterato del tipo $\displaystyle\int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx$ con $a \le b$ e $\alpha(x) \le \beta(x)$ è l'integrale doppio di f su $D = \{x \in [a,b],\ \alpha(x) \le y \le \beta(x)\}$.

!! Un integrale iterato non è sempre un integrale doppio. Ad esempio $\displaystyle\int_{-1}^{1} \left\{ \int_x^{2x} y\,dy \right\} dx$ non è l'integrale doppio di y su una regione D. !!

Si avrebbe infatti:

$$D = \{ (x,y) : x \in [-1,1],\ \underbrace{x \le y \le 2x}_{\Downarrow \atop x \ge 0} \} = \{ (x,y) : x \in [0,1],\ x \le y \le 2x \}$$

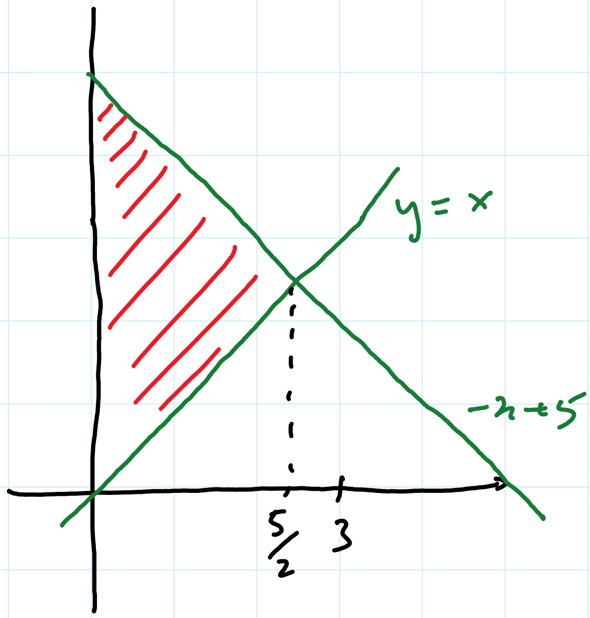
$$\Rightarrow \int_D y\,dx\,dy = \int_0^1 \left\{ \int_x^{2x} y\,dy \right\} dx :$$ non è l'integrale iterato iniziale!

ATTENZIONE AGLI INGANNI.

$D = \{ (x,y) : x \in [0,3] \quad x \le y \le -x+5 \}$.

$$\int_D f(x,y)\,dx\,dy \overset{?}{=} \int_0^3 \int_x^{-x+5} f(x,y)\,dy\,dx$$

$$x \le y \le -x+5 \Rightarrow x \le -x+5 \Rightarrow 2x \le 5 \qquad x \le \frac{5}{2}$$

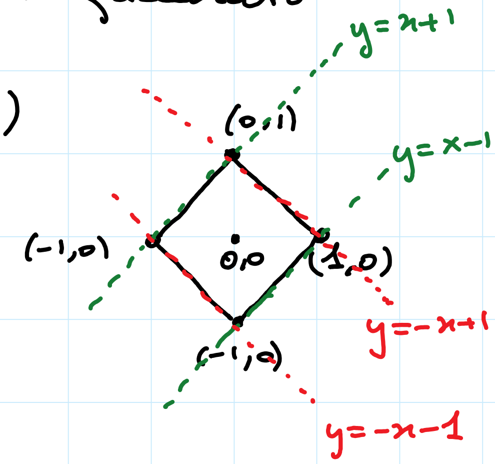


$$\int_D f(x,y)\,dx\,dy$$
$$\shortparallel$$
$$\int_0^{5/2} \left\{ \int_x^{-x+5} f(x,y)\,dy \right\} dx .$$

ESEMPIO. Calcoliamo

$$\int_D (x^2-y^2)^2\,dx\,dy \quad \text{dove } D \text{ è il quadrato}$$

di vertici $(\pm 1, 0)$, $(0, \pm 1)$



$$(x^2-y^2)^2 = (y^2-x^2)^2 = (y-x)^2(y+x)^2$$

$$u := x+y \qquad v := y-x \iff \begin{cases} x = \dfrac{u-v}{2} \\ y = \dfrac{u+v}{2} \end{cases}$$

$$\varphi(u,v) := \left(\frac{u-v}{2}, \frac{u+v}{2}\right). \qquad (x,y) \in D \iff \begin{cases} -x-1 \le y \le -x+1 \\ x-1 \le y \le x+1 \end{cases}$$

$$\iff \begin{cases} -1 \le y+x \le 1 \\ -1 \le y-x \le 1 \end{cases}$$

Se $\varphi(u,v) = (x,y)$: $(x,y) \in D \iff -1 \le u \le 1 \quad -1 \le v \le 1$

$-1 \le y-x \le 1$

Se $\varphi(u,v)=(x,y)$: $\quad (x,y)\in D \Leftrightarrow -1\le u\le 1, \; -1\le v\le 1$

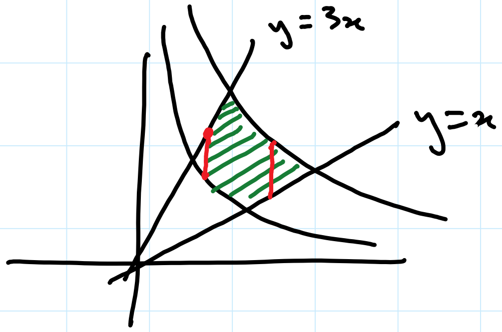
$$\varphi'(u,v)=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad \det \varphi'(u,v)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\int_D (y-x)^2(y+x)^2 = \int_{[-1,1]\times[-1,1]} v^2u^2\,\frac{1}{2}\,du\,dv = \frac{1}{2}\int_{-1}^{1}u^2\,du\int_{-1}^{1}v^2\,dv$$

$$=\frac{1}{2}\,2\int_0^1 u^2\,du\;\;2\int_0^1 v^2\,dv$$

$$=2\left(\frac{1}{3}\right)^2=\frac{2}{9}.$$

ESERCIZIO. $\displaystyle\int_D xy\,dx\,dy \qquad D=\left\{(x,y):\; x>0,\, y>0 \;\; x\le y\le 3x,\; \frac{1}{x}\le y\le\frac{3}{x}\right\}$



$$=\Big\{(x,y):\; x>0,\; y>0:\; \underbrace{1\le \frac{y}{x}\le 3}_{u},\;\; \underbrace{1\le xy\le 3}_{v}\Big\}$$

Poniamo $u=\dfrac{y}{x}$ e $v=xy$.

Siano $u>0$, $v>0$. Risolviamo $u=\dfrac{y}{x}$, $v=xy$.

$u=\dfrac{y}{x} \Rightarrow y=xu$; $\quad v=xy \Rightarrow v=x^2u \Rightarrow x^2=v/u$

Esiste unica $x>0$: $x=\sqrt{v/u}$. Quindi $y=\dfrac{\sqrt{v}}{\sqrt{u}}\,u=\sqrt{uv}$.

$Q=\{(x,y): x>0, y>0\}$

$(x,y)\longmapsto \Big(\underset{u}{\overset{\|}{\tfrac{y}{x}}},\, \underset{v}{\overset{\|}{xy}}\Big)\in Q$ è biiettiva.

Q

φ

Q

$$(u,v) \xmapsto{\ \varphi\ } \left(\frac{\sqrt{v}}{\sqrt{u}}, \sqrt{uv}\right) : \varphi \text{ è biiettiva. } \varphi \text{ è } C^1,$$

$$\varphi'(u,v) = \begin{pmatrix} -\frac{1}{2} v^{\frac{1}{2}} u^{-\frac{3}{2}} & \frac{1}{2} v^{-\frac{1}{2}} u^{-\frac{1}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{pmatrix}$$

$$\det \varphi'(u,v) = -\frac{1}{4} v^0 u^{-1} - \frac{1}{4} v^0 u^{-1} = -\frac{1}{2} u^{-1}$$

C. di variabile

$$\int_D xy \, dx\, dy = \int_{[1,3]\times[1,3]} v \left|-\frac{1}{2}u^{-1}\right| du\, dv = \frac{1}{2} \int_{[1,3]\times[1,3]} \frac{v}{u} \, du\, dv$$

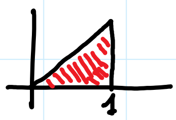
$$= \frac{1}{2} \int_1^3 \frac{1}{u} du \int_1^3 v \, dv = \frac{1}{2} \left[\lg u\right]_1^3 \left[\frac{1}{2} v^2\right]_1^3$$

$$= \frac{1}{2} \lg 3 \ \frac{1}{2} (8) = 2 \lg 3 .$$

! Il passaggio in coordinate polari non conduce sempre ad una integrazione su intervalli!

ESEMPIO. Calcolare

$$\int_D \frac{x}{x^2+y^2} \, dx\, dy \qquad D = \{(x,y) : x \in [0,1], \ 0 \le y \le x\}$$



Utilizziamo per esercizio le coordinate polari.

Si osservi che, posto $x = \rho \cos t$, $y = \rho \sin t$ con $\rho \ge 0$ e $t \in [0, 2\pi]$

si ha

$(\rho\cos t, \rho\sin t) \in D \iff \rho\cos t \in [0,1]$ e $0 \le \rho\sin t \le \rho\cos t$

$\iff \rho \in \left[0, \frac{1}{\cos t}\right], \quad t \in \left[0, \frac{\pi}{4}\right].$

Si ha pertanto

$$\int_D \frac{x}{x^2+y^2}\,dx\,dy = \int_0^{\pi/4} \int_0^{\frac{1}{\cos t}} \frac{\rho\cos t}{\rho^2}\,\rho\,d\rho\,dt$$

$$= \int_0^{\pi/4} \int_0^{\frac{1}{\cos t}} \cos t\,d\rho\,dt = \int_0^{\pi/4} (\cos t)\cdot\frac{1}{\cos t}\,dt$$

$$= \pi/4 .$$

ESERCIZIO. Calcolare $\displaystyle\int_0^{\sqrt2}\left\{\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\right\}dy$

Si tratta dell'integrale doppio di $\frac{1}{1+x^2+y^2}$ nel dominio:

$$D=\{(x,y):\ y\in[0,\sqrt2],\quad y\le x\le\sqrt{4-y^2}\}$$

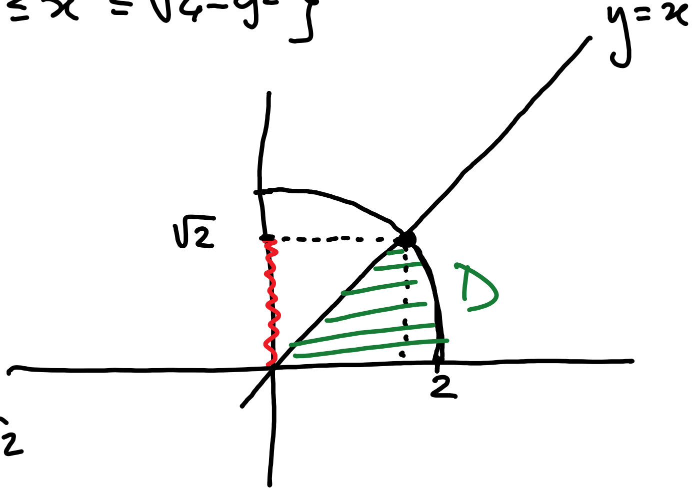
$(0\le)\, x\le\sqrt{4-y^2}$

$x^2\le 4-y^2$

$x^2+y^2\le 4$

$\{y=x\}\cap\{x^2+y^2=4\}$

$\begin{cases} y=x \\ x^2+y^2=4\end{cases} \iff \begin{cases} y=x \\ 2x^2=4\end{cases} \iff y=x=\sqrt2$



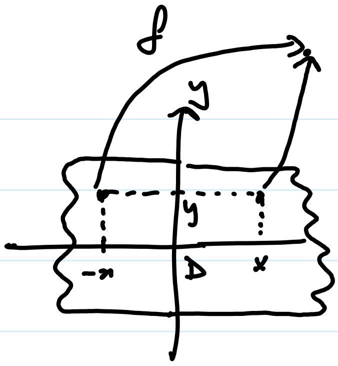
$$\int_0^{\sqrt2}\left\{\int_y^{\sqrt{4-y^2}}\frac{1}{1+x^2+y^2}\,dx\right\}dy=\int_D \frac{1}{1+x^2+y^2}\,dx\,dy$$

$$=\int_{\substack{0\le\rho\le2\\ t\in[0,\pi/4]}} \frac{1}{1+\rho^2}\,\rho\,d\rho\,dt=\left(\int_0^2\frac{\rho}{1+\rho^2}\,d\rho\right)\left(\int_0^{\pi/4}1\,dt\right)$$

$$=\left[\frac12\lg(1+\rho^2)\right]_0^2\frac{\pi}{4}=\frac{\pi}{8}\lg5$$

SIMMETRIE.

$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$



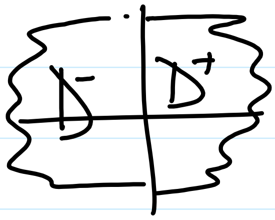
D simmetrico rispetto all'asse y

$(x,y) \in D \Rightarrow (-x,y) \in D$

1) $f(-x,y) = -f(x,y) \Rightarrow \int_D f(x,y)\,dx\,dy = 0 \quad (1)$

2) $f(-x,y) = f(x,y) \Rightarrow \int_D f(x,y)\,dx\,dy = 2\int_{D\cap\{x>0\}} f(x,y)\,dx\,dy.$

Proviamo (1): $\int_D f(x,y)\,dx\,dy$



$D^- = \{(x,y) \in D : x \le 0\}$

$D^+ = \{(x,y) \in D : x \ge 0\}$

$$\int_D f(x,y)\,dx\,dy = \boxed{\int_{D^-} f(x,y)\,dx\,dy} + \int_{D^+} f(x,y)\,dx\,dy$$

$\int_{D^-} f(x,y)\,dx\,dy \qquad$ Poniamo $\begin{cases} u = -x \\ v = y \end{cases} \Leftrightarrow \begin{cases} x = -u \\ y = v \end{cases}$

Jac: $(u,v) \longmapsto (-u,v)$: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$|\det \text{Jac} \dots| = 1$

$$\int_{D^-} f(x,y)\,dx\,dy = \int_{D^+} \underbrace{f(-u,v)}_{=-f(u,v)}\, 1\,du\,dv = -\int_{D^+} f(u,v)\,du\,dv$$

$$\Rightarrow \int_D f = \int f + \int f = -\int f + \int f = 0.$$

$$\Rightarrow \int_D f = \int_{D^-} f + \int_{D^+} f = -\int_{D^+} f + \int_{D^+} f = 0.$$

BARICENTRO DI UN INSIEME $D \subseteq \mathbb{R}^2$:

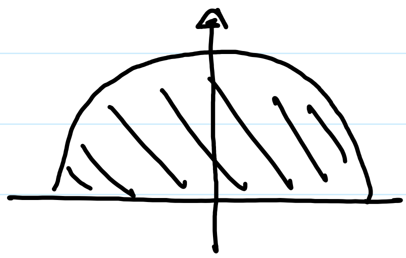
È il punto $(\bar{x}, \bar{y})$ definito da

$$\bar{x} = \frac{\int_D x \, dx\,dy}{\text{Area}(D)}, \quad \bar{y} = \frac{\int_D y \, dx\,dy}{\text{Area}(D)}$$

$$\text{Area}(D) := \int_D 1 \, dx\,dy$$

ESEMPIO. Baricentro del semidisco $B^+(0,R] = D$

$\{(x,y): \ y \geq 0, \ \ x^2+y^2 \leq R^2\}$.



$\bar{x} = 0$:

$$\bar{x} = \frac{\int_D x \, dx\,dy}{\text{Area}\, D} \qquad -x = -(x) \qquad \int_D x \, dx\,dy = 0.$$

$$\bar{y} = \frac{\int_D y \, dx\,dy}{\text{Area}\, D} = \frac{\int_D y \, dx\,dy}{\frac{\pi}{2} R^2} = 2 \, \frac{\int_D y \, dx\,dy}{\pi R^2}.$$

$$\int_D y \, dx\,dy = \int_{\substack{0 \leq \rho \leq R \\ 0 \leq t \leq \pi}} (\rho \sin t) \rho \, d\rho\, dt = \left(\int_0^R \rho^2 d\rho\right)\left(\int_0^\pi \sin t \, dt\right)$$

$$= \frac{1}{3} R^3 \left[-\cos t\right]_0^\pi = \frac{2}{3} R^3$$

$$\bar{y} = 2\,\frac{\frac{2}{3}R^3}{\pi R^2} = \frac{4}{3}\,\frac{R^3}{\pi R^2} = \frac{4}{3\pi}R$$

ESERCIZIO Calcolare l'area della regione del piano delimitata da:

$$\frac{x^2}{4} \le y \le x^2 \quad e \quad 2 \le xy \le 5$$

È un sottoinsieme di $\{x>0, y>0\}$.

$$\frac{1}{4} \le \frac{y}{x^2} \le 1 \qquad 2 \le xy \le 5$$

Poniamo $u = \frac{y}{x^2}$, $v = xy$

$$(x,y) \xmapsto{\psi} \left(\frac{y}{x^2}, xy\right)$$

$]0,+\infty[\times]0,+\infty[\longrightarrow]0,+\infty[\times]0,+\infty[$ invertibile e l'inversa ϕ è $\mathcal{C}^1$ e $\det \phi' \neq 0$.

Fissati $u>0, v>0$ studiamo l'equazione

$$(u,v) = \psi(x,y) \iff \begin{cases} u = y/x^2 & (1) \\ v = xy & (2) \end{cases}$$

$(1) \Rightarrow y = ux^2 \quad (2) \Rightarrow v = x(ux^2) = x^3 u \Rightarrow x^3 = \frac{v}{u} \Rightarrow x = \frac{v^{1/3}}{u^{1/3}}$

$$y = u x^2 = u \cdot \frac{v^{2/3}}{u^{2/3}} = u^{1/3} v^{2/3}$$

Poniamo $\begin{cases} x = v^{1/3} u^{-1/3} \\ y = u^{1/3} v^{2/3} \end{cases}$ $\phi(u,v) = (u^{-1/3} v^{1/3}, u^{1/3} v^{2/3})$.

ϕ è biiettiva (è inversa di ψ), è $\mathcal{C}^1$.

Calcoliamo $\det \operatorname{Jac} \phi(u,v)$:

$$\operatorname{Jac} \phi(u,v) = \begin{pmatrix} -\frac{1}{3} u^{-4/3} v^{1/3} & \frac{1}{3} u^{-1/3} v^{-2/3} \\ \ldots \end{pmatrix}$$

$$\mathrm{Jac}\,\phi(u,v) = \begin{pmatrix} \ldots & \ldots \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{pmatrix}$$

$$\det \mathrm{Jac}\,\phi(u,v) = -\frac{2}{9}u^{-1}v^{0} - \frac{1}{9}u^{-1}v^{0} = -\frac{1}{3}u^{-1} \neq 0$$

$$R = \left\{ (x,y) : \frac{x^2}{4} \le y \le x^2, \quad 2 \le xy \le 5 \right\}$$

$$: \quad \underbrace{\frac{1}{4} \le \frac{y}{x^2} \le 1}_{u}, \quad \underbrace{2 \le xy \le 5}_{v} \Big\}$$

$$\phi(u,v) \in R \iff u \in \left[\tfrac{1}{4}, 1\right], \quad v \in [2,5]$$

$$\text{Area } R = \int_R 1\,dx\,dy = \int_{[\frac{1}{4},1]_u \times [2,5]_v} \frac{1}{3u}\,du\,dv = \frac{1}{3}\left(\int_{1/4}^{1} \frac{1}{u}\,du\right)\left(\int_2^5 1\,dv\right)$$

$$= \frac{1}{3}\left[\lg u\right]_{1/4}^{1} \cdot 3 = \lg 4.$$