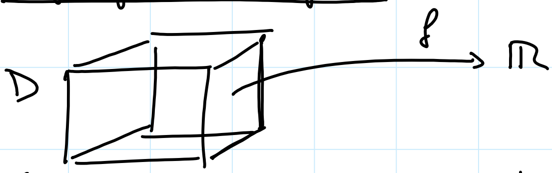
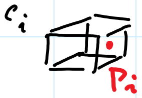
INTEGRALE IN DIMENSIONE 3.

Definizione di integrale.



Suddividiamo il cubo di partenza in unione di cubi $\{C_1, \dots, C_m\}$



. Scegliamo $p_i \in C_i$, e calcoliamo

$f(p_1)\,\mathrm{Vol}(C_1) + \dots + f(p_m)\,\mathrm{Vol}(C_m)$.

Sotto adeguate condizioni (es: f continua) tali somme convergono, al tendere a 0 dei lati dei cubi C_i, ad un numero che si chiama l'integrale su D di f, si indica con

$$\int_D f(x)\,dx = \int_D f(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3$$

OSS: se $f = 1$ è $\int_D 1\,dx$ = Volume di D

Si estende poi, come su $\mathbb{R}^2$, la nozione di integrale a funzioni definite su insiemi limitati esattamente come si fa in $\mathbb{R}^2$.

DEF. Se $D \subseteq \mathbb{R}^3$ è limitato si pone $\mathrm{Vol}(D) = \int_D 1\,dx$.

PROPRIETÀ DELL'INTEGRALE.

- Se $D_1 \cap D_2 = \emptyset$ $\int_{D_1 \cup D_2} f = \int_{D_1} f + \int_{D_2} f$;
- $\int_D f+g = \int_D f + \int_D g$.

INTEGRAZIONE PER FILI PARALLELI AD UN ASSE COORDINATO

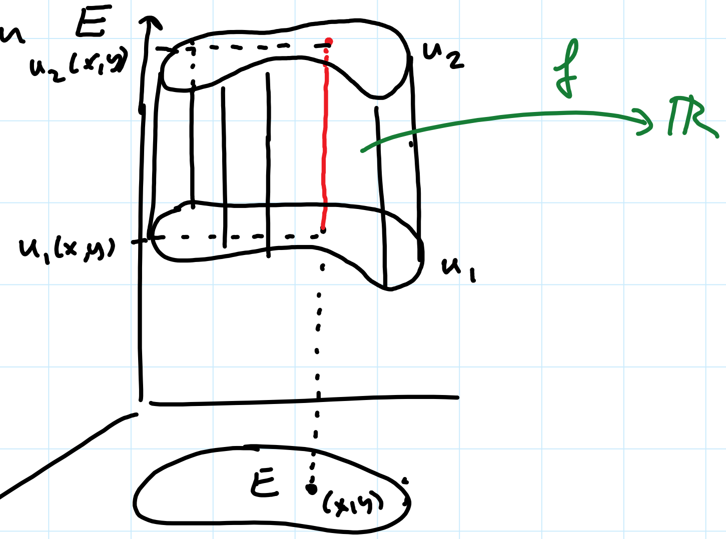
$D = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in E \subseteq \mathbb{R}^2 ;$

$u_1(x,y) \le z \le u_2(x,y) \}$

Dominio semplice risp. a (x,y)

OSS: $E =$ proiezione di D su x,y

con $\boxed{u_1 \le u_2}$ su E



$f : D \to \mathbb{R}$

$$\int_D f(x,y,z)\,dx\,dy\,dz$$

integrale triplo di f su D

$$= \int_E \underbrace{\left\{ \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right\}}_{\text{Funzione di } (x,y)} dx\,dy$$

OSS: Se $f = 1$ $\quad \int_D 1\,dx\,dy\,dz =: \mathrm{Vol}(D)$.

La formula dice che $\mathrm{Vol}(D) = \int_E \underbrace{\mathrm{Lungh}\left[u_1(x,y), u_2(x,y)\right]}_{u_2(x,y) - u_1(x,y)} dx\,dy$

ES. $D = \{ (x,y,z) : x^2+y^2 \le 3, \ \overset{u_1}{0} \le z \le \overset{u_2}{3+y} \}$.

- sugg: osservare che $3+y \ge 0 \quad \forall (x,y) : x^2+y^2 \le 3$

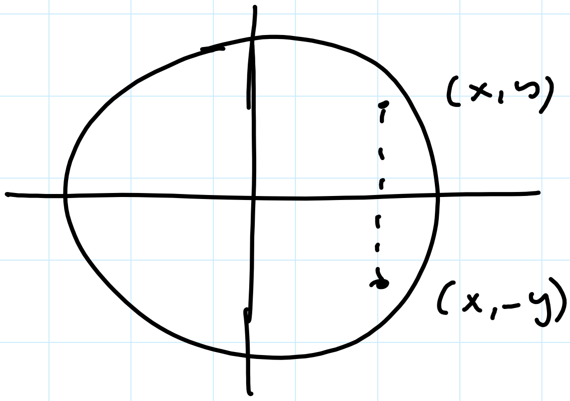
Calcolo del Volume di D.

$$\mathrm{Vol}(D) = \int 1\,dx\,dy\,dz = \int \left\{ \int^{3+y} dz \right\} dx\,dy = \int 3+y\,dx\,dy$$

$$\mathrm{Vol}(D) = \int_D 1 \, dx\,dy\,dz = \int_{x^2+y^2\le 3} \left\{ \int_0^{3+y} dz \right\} dx\,dy = \int_{x^2+y^2\le 3} 3+y \; dx\,dy$$

$$= \int_{x^2+y^2\le 3} 3 \, dx\,dy + \int_{x^2+y^2\le 3} y \, dx\,dy$$

$$= 3 \times \pi (\sqrt{3})^2 + 0 \, .$$

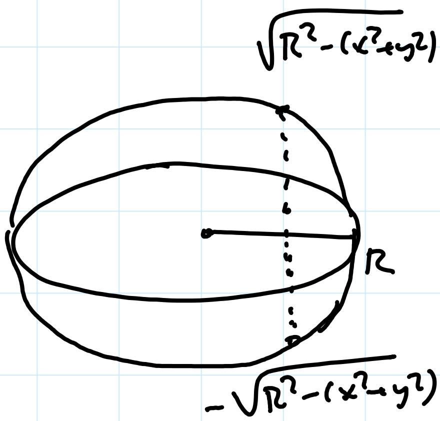


ESERCIZIO. Volume della palla di raggio $R>0$

$$B_R = \{ (x,y,z) : x^2+y^2+z^2 \le R^2 \}$$
$$= \{ (x,y,z) : x^2+y^2 \le R^2 \; ; \; -\sqrt{R^2-(x^2+y^2)} \le z \le \sqrt{R^2-(x^2+y^2)} \}$$

$$\mathrm{Vol}(B_R) = \int_{B_R} 1 \, dx\,dy\,dz$$

$$= \int_{\underbrace{x^2+y^2\le R^2}_{\subseteq \mathbb{R}^2}} \int_{-\sqrt{R^2-(x^2+y^2)}}^{\sqrt{R^2-(x^2+y^2)}} 1 \, dz \quad dx\,dy$$



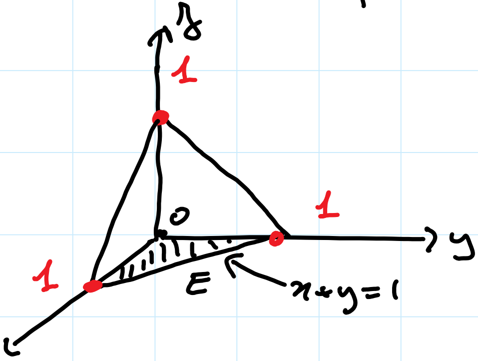
$$= \int_{x^2+y^2\le R^2} 2\sqrt{R^2-(x^2+y^2)} \, dx\,dy = 2 \int_{\substack{\rho\in[0,R] \\ t\in[0,2\pi]}} \rho\sqrt{R^2-\rho^2} \, d\rho\, dt$$

$$= 2\left(\int_0^R \rho \sqrt{R^2-\rho^2}\, d\rho \right) 2\pi = 4\pi \left[-\frac{1}{3} (R^2-\rho^2)^{\frac{3}{2}} \right]_0^R = \frac{4\pi R^3}{3}$$

ESERCIZIO. $T = \{(x,y,z) : 0 \le x,\ 0 \le y,\ 0 \le z;\ x+y+z \le 1\}$

$z \le 1-x-y$ $\qquad$ $z = 1-x-y$ è un piano.

$T = \{(x,y,z) : x \ge 0,\ y \ge 0,\ 0 \le z \le 1-x-y\}$



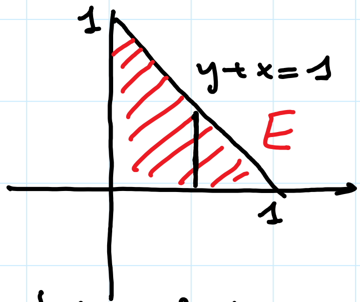
Proiezione di T nel piano xy:

$\{(x,y) \in \mathbb{R}^2 : \exists z \text{ e } (x,y,z) \in T$

$: \exists z \text{ e } \begin{cases} x \ge 0 \\ y \ge 0 \\ 0 \le z \le 1-x-y \end{cases} \Big\} = \{(x,y) : x \ge 0,\ y \ge 0,\ x+y \le 1$

$$\mathrm{Vol}(T) = \int_T 1\, dx\, dy\, dz$$

$$= \int_E \left\{ \int_0^{1-x-y} 1\, dz \right\} dx\, dy = \int_E 1-x-y\ dx\, dy$$



$$= \int_0^1 \left\{ \int_0^{1-x} (1-x-y)\, dy \right\} dx$$

$$= \int_0^1 \left[y - xy - \tfrac{1}{2} y^2 \right]_{y=0}^{1-x} dx = \int_0^1 (1-x) - x(1-x) - \tfrac{1}{2}(1-x)^2\, dx$$

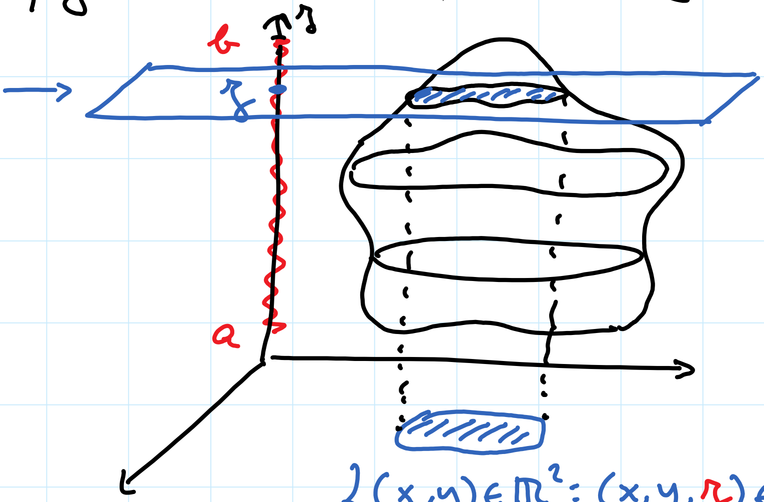
$$= \int_0^1 \underbrace{1-2x+x^2}_{(x-1)^2} - \tfrac{1}{2}(x-1)^2\, dx = \frac{1}{2}\int_0^1 (x-1)^2 dx = \frac{1}{2}\int_0^1 u^2\, du = \frac{1}{6}$$

INTEGRAZIONE PER FETTE PARALLELE AD

UN PIANO COORDINATO.

$D \subseteq \mathbb{R}^3$ e sia $[a,b]$ la proiezione di D sull'asse z: $\{z \in \mathbb{R} : \exists (x,y) : (x,y,z) \in D\}$

Punti di D a quota z



$\{(x,y) \in \mathbb{R}^2 : (x,y,z) \in D\} = D_z$

z-sezione di D

Se $f: D \to \mathbb{R}$

$$\int_D f \, dxdydz$$

$$\parallel$$

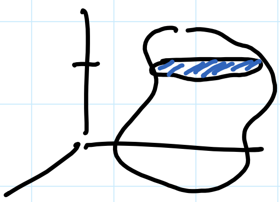
$$\int_a^b \underbrace{\left\{ \int_{D_z} f(x,y,z)\, dxdy \right\}}_{\text{integrale doppio che dipende solo da } z} dz$$

Se $f = 1$ $\quad \int_D 1 \, dxdydz = Vol(D)$.

La formula porge

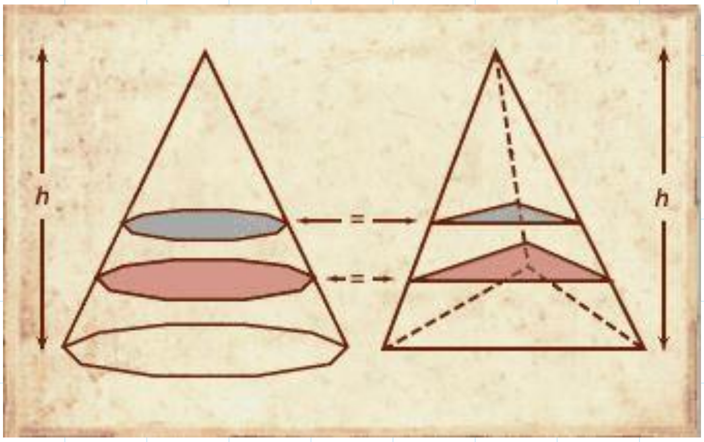
$$Vol(D) = \int_a^b \left\{ \int_{D_z} 1 \, dxdy \right\} dz = \int_a^b Area(D_z)\, dz$$

Si integrano le aree delle fette



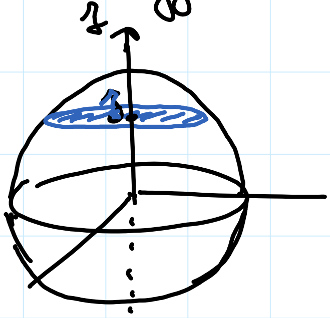
$\rightsquigarrow$ Principio di Bonaventura - Cavalieri

Due solidi con sezioni di uguale area hanno lo stesso volume.



ESEMPIO. Volume della palla di raggio R.

$$\mathrm{Vol}(B_R) = \int_{-R}^{R} \mathrm{Area}\,(B_R)_z \, dz$$



$$D_z = \{ (x,y) \in \mathbb{R}^2 : \quad (x,y,z) \in D \}$$

$$(B_R)_z = \{ (x,y) \in \mathbb{R}^2 : \quad x^2+y^2+z^2 \le R^2 \} = \{ (x,y) : \ x^2+y^2 \le R^2 - z^2 \}$$

$= $ Disco centro 0, raggio $\sqrt{R^2 - z^2}$

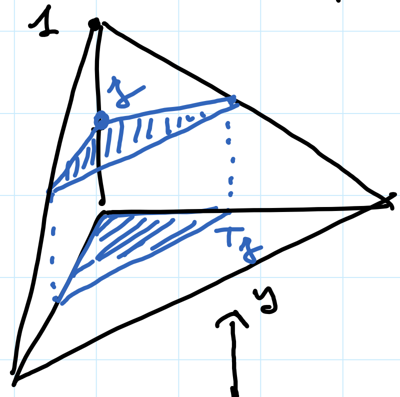
$$\mathrm{Vol}(B_R) = \int_{-R}^{R} \mathrm{Area}((B_R)_z)\, dz = \int_{-R}^{R} \pi (R^2 - z^2)\, dz$$

$$= 2\pi \int_0^R (R^2 - z^2)\, dz = 2\pi \left[R^2 z - \frac{1}{3} z^3 \right]_0^R$$

$$= 2\pi \left(R^3 - \frac{R^3}{3} \right) = 2\pi \cdot \frac{2R^3}{3} = \frac{4\pi}{3} R^3$$

ESEMPIO. $T=\{(x,y,z): x\ge 0, y\ge 0, z\ge 0, \ x+y+z\le 1\}$.

Calcolo di Vol(T) integrando per fette parallele al piano xy



Proiezione sull'asse z: $[0,1]$.

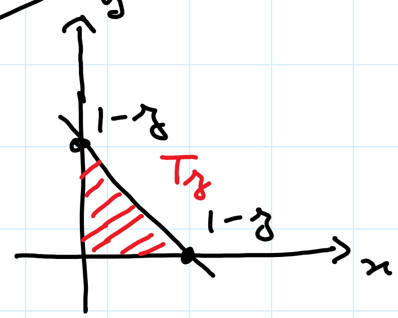
Fissato $z\in[0,1]$

$T_z=\{(x,y): (x,y,z)\in T\}$

$\downarrow$

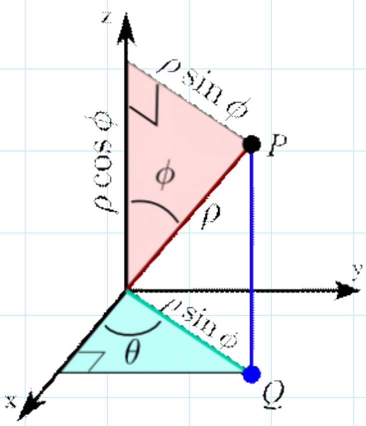
$x\ge 0, y\ge 0, \ x+y+z\le 1$

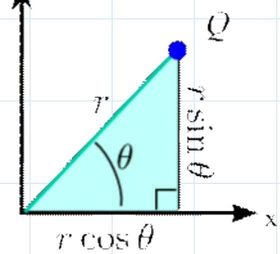
$x+y\le 1-z$



$\text{Area}(T_z)=\frac{1}{2}(1-z)^2$

$$\text{Vol } T=\int_0^1 \text{Area}(T_z)\,dz=\int_0^1 \frac{1}{2}(z-1)^2\,dz=\frac{1}{2}\int_0^1 u^2\,du=\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{6}$$


z
$\rho\sin\phi$
$\rho\cos\phi$
P
ϕ
ρ
y
$\rho\sin\phi$
θ
x
Q


y
Q
r
$r\sin\theta$
θ
x
$r\cos\theta$