SUPERFICIE PARAMETRICA

DEF. Una superficie parametrica in $\mathbb{R}^3$ è una FUNZIONE $p : \underset{\text{CHIUSO}}{D \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$ di classe $\mathcal{C}^1$

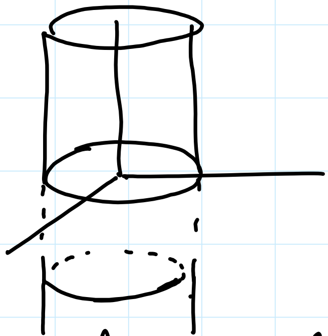
$$p : \underbrace{(u, v)}_{\text{parametri}} \longmapsto (p_1(u,v), p_2(u,v), p_3(u,v))$$

p_1, p_2, p_3 $\mathcal{C}^1$ (in un aperto contenente D)

OSS: ricordare il concetto di curva: FUNZIONE
$\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$ o $\mathbb{R}^2$

Il sostegno della superficie parametrica è l'insieme
$p(D) = \{ p(u,v) : (u,v) \in D \}$.

ESEMPIO. $R > 0$ $\quad (R\cos t, R\sin t, z)$
$t \in [0, 2\pi]$, $z \in [0, h]$
$(R\cos t)^2 + (R\sin t)^2 = R^2 \Rightarrow (R\cos t, R\sin t) \in$ cerchio di raggio R.



Tale insieme è il sostegno della superficie param.
$(t, z) \longmapsto (R\cos t, R\sin t, z)$.

ESEMPIO (Superficie cartesiana)
$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ è $\mathcal{C}^1$
$p(x,y) = (x, y, f(x,y)) \in \mathbb{R}^3 \quad \forall (x,y) \in D$
Sostegno di p = Grafico di f.

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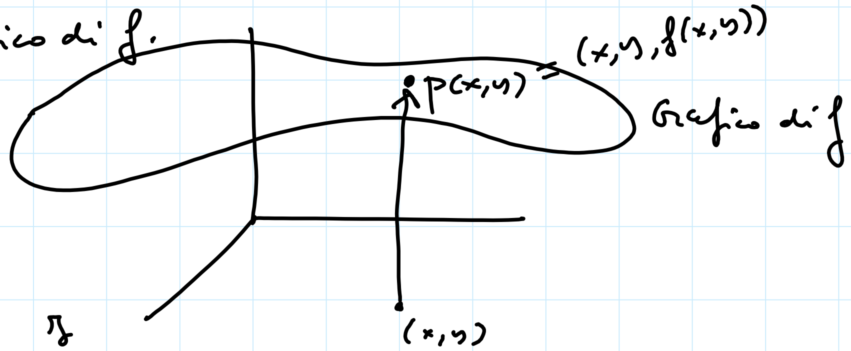
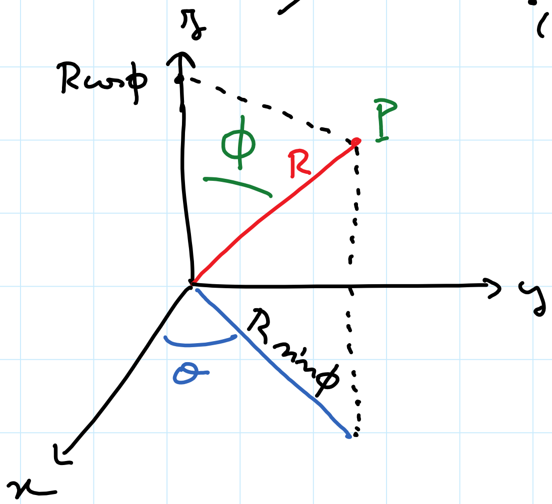


ESEMPIO.

R fissato



$$p(\phi,\theta) = (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

$\phi \in [0,\pi]$, $\theta \in [0,2\pi]$ $\quad p$ è superficie parametrica;

sostegno di p = Sfera di raggio R.

ESEMPIO: L'emisfero Nord come sostegno di una superficie cartesiana:

$R>0$, sfera di raggio R: $\{(x,y,z) : x^2+y^2+z^2=R^2\}$;

$z \geq 0 \Rightarrow z = \sqrt{R^2-(x^2+y^2)}$,

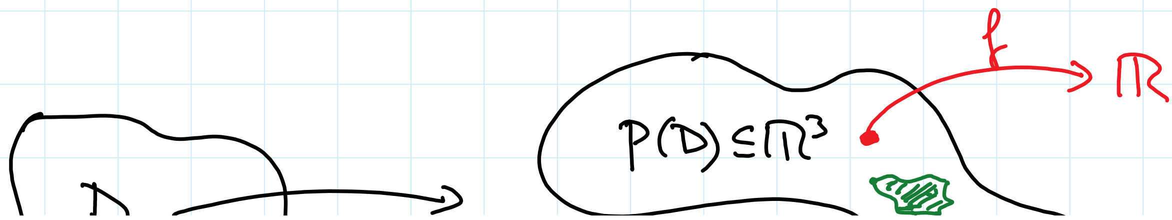
con $x^2+y^2 \leq R^2$

l'emisfero "Nord" è il

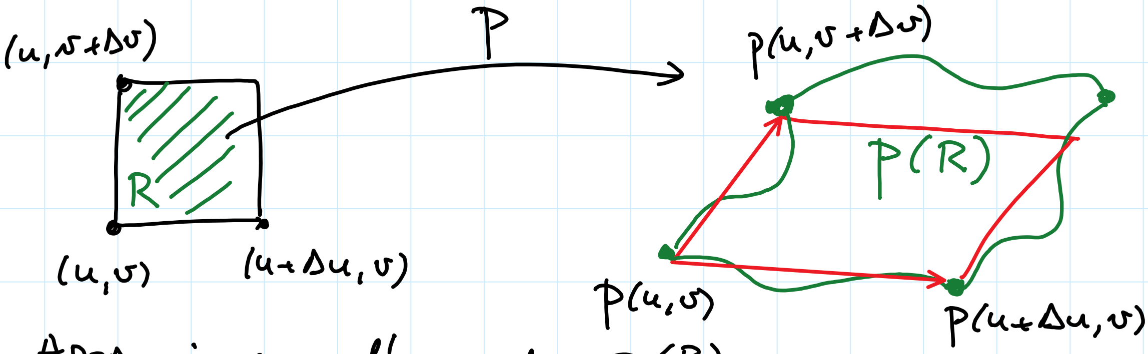
sostegno di $p(x,y) = (x,y,\sqrt{R^2-(x^2+y^2)})$

INTEGRALI SUPERFICIALI

Sia $p : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ superficie parametrica.



Vediamo come si modificano localmente le aree dei rettangoli in D tramite p



Approssimare l'area di $p(R)$ con l'area del parallelogramma generato dai vettori $\underbrace{p(u+\Delta u, v) - p(u,v)}_{\approx\, \partial_u p(u,v)\,\Delta u}$; $\underbrace{p(u, v+\Delta v) - p(u,v)}_{\approx\, \partial_v p(u,v)\,\Delta v}$.

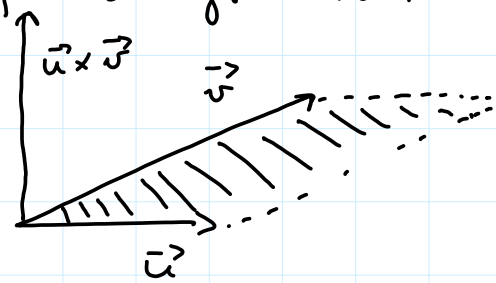
$$\left[\text{Se } p = (p^1, p^2, p^3) \qquad \partial_u p = (\partial_u p^1, \partial_u p^2, \partial_u p^3)\right]$$

e vale $\| \partial_u p(u,v)\Delta u \times \partial_v p(u,v)\Delta v \|$

$$= \underbrace{\| \partial_u p(u,v) \times \partial_v p(u,v) \|}_{\text{elemento d'area (bidimensionale) di } p} \; \underbrace{|\Delta u|\,|\Delta v|}_{\text{Area di } R}$$

Ripasso: area di un parallelogramma.

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\,|\sin(\vec{u},\vec{v})|$$



$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$$

$\longrightarrow \big| \ \vec{e}_1 \qquad \vec{e}_2 \qquad \vec{e}_3 \ \big|$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{e}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{e}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{e}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

L'area del parallelepipedo generato da $\vec{u}, \vec{v}$ è

$$\boxed{\sqrt{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2}}$$

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

ESERCIZIO. $\vec{u} = (1, -3, 4)$ $\quad \vec{v} = (-1, 2, -5)$

Area del parallelogramma generato da $\vec{u}$ e $\vec{v}$.

$$\begin{pmatrix} 1 & -3 & 4 \\ -1 & 2 & -5 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} = 2-3 = -1 \longrightarrow (-1)^2 = 1$$

$$\begin{vmatrix} 1 & 4 \\ -1 & -5 \end{vmatrix} = -5+4 = -1 \longrightarrow (-1)^2 = 1$$

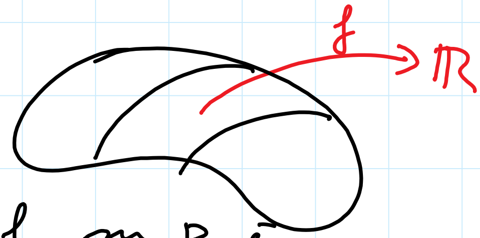
$$\begin{vmatrix} -3 & 4 \\ 2 & -5 \end{vmatrix} = 15-8 = 7 \longrightarrow 7^2 = 49$$

Area: $\sqrt{49+1+1} = \sqrt{51}. = \|\vec{u} \times \vec{v}\|$

DEF. La misura superficiale (o "area") della superficie P è

$$\int_D \|\partial_u p(u,v) \times \partial_v p(u,v)\| \, du\, dv$$

Se $f : p(D) \longrightarrow \mathbb{R}$



l'integrale superficiale di f su p è

$$\int_p f \, d\sigma_p = \int_p f(x,y,z) \, d\sigma_p(x,y,z)$$

$$= \int_D f(p(u,v)) \, \|\partial_u p(u,v) \times \partial_v p(u,v)\| \, du\, dv.$$

Come si calcola l'elemento d'area:

$$\partial_u p = (\partial_u p_1, \partial_u p_2, \partial_u p_3)$$
$$\partial_v p = (\partial_v p_1, \partial_v p_2, \partial_v p_3)$$

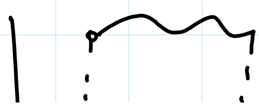
$\|\partial_u p \times \partial_v p\|^2 = \sum$ quadrati dei determinanti 2×2 della matrice $[\partial_u p, \partial_v p]$.

SUPERFICIE CARTESIANA.

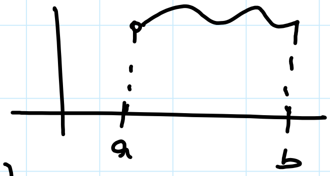
$p(x,y) = (x, y, f(x,y))$ $\qquad f : D \subseteq \mathbb{R}^2 \xrightarrow{\mathcal{C}^1} \mathbb{R}$

$$\boxed{\begin{array}{l} \text{Area di } p \ (\text{cartesiana } p(x,y) = (x,y,f(x,y))_{(x,y)\in D}) \text{ è} \\ \displaystyle\int_D \sqrt{1 + \|\nabla f(x,y)\|^2} \, dx\, dy \end{array}}$$

OSS: Ricordare l'analogo risultato visto sulle curve.
Sia $f : [a,b] \longrightarrow \mathbb{R}$ $\mathcal{C}^1$ e consideriamo
la curva $\gamma(t) = (t, f(t))$



· Sia $f : [a,b] \to \mathbb{R}$ $\mathcal{C}^1$ e consideriamo la curva $\gamma(t) = (t, f(t))$



Lunghezza di γ

$$= \int_a^b \|\gamma'(t)\| dt \qquad \gamma'(t) = (1, f'(t))$$

$$\|\gamma'(t)\| = \sqrt{1 + (f'(t))^2}$$

$$\Rightarrow \text{Lungh}(\gamma) = \int_a^b \sqrt{1 + f'(t)^2}\, dt$$

$$p(x,y) = (x, y, f(x,y)) \qquad \partial_x p(x,y) = (1, 0, \partial_x f(x,y))$$

$$\partial_y p(x,y) = (0, 1, \partial_y f(x,y))$$

$$\|\partial_x p \times \partial_y p\|^2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2 + \begin{vmatrix} 1 & \partial_x f \\ 0 & \partial_y f \end{vmatrix}^2 + \begin{vmatrix} 0 & \partial_x f \\ 1 & \partial_y f \end{vmatrix}^2$$

$$= 1 + (\partial_y f)^2 + (\partial_x f)^2 = 1 + \|\nabla f(x,y)\|^2$$

$$\Rightarrow \boxed{\|\partial_x p \times \partial_y p\|(x,y) = \sqrt{1 + \|\nabla f(x,y)\|^2}}$$

ES. Area della semisfera di raggio R.

$$f(x,y) = \sqrt{R^2 - x^2 - y^2}, \quad \boxed{x^2 + y^2 \le R^2}$$

$$\int_{x^2+y^2 \le R^2} \sqrt{1 + \|\nabla f(x,y)\|^2}\, dx\, dy = \text{Area di metà sfera.}$$

$$\partial_x f(x,y) = \frac{-2x}{2\sqrt{R^2 - x^2 - y^2}} = \frac{-x}{\sqrt{R^2 - (x^2+y^2)}} \; ; \; \partial_y f(x,y) = \frac{-y}{\sqrt{R^2 - (x^2+y^2)}}$$

$$\|\nabla f(x,y)\|^2 = \frac{x^2}{R^2 - (x^2+y^2)} + \frac{y^2}{R^2 - (x^2+y^2)} \quad \Big| \quad 1 + \|\nabla f(x,y)\|^2 = \frac{R^2}{R^2 - (x^2+y^2)}$$

$$\text{Area(Metà sfera)} = \int_{x^2+y^2 \le R^2} \sqrt{\frac{R^2}{R^2 - (x^2+y^2)}}\, dx\, dy = \int_{x^2+y^2 \le R^2} \frac{R}{\sqrt{R^2 - (x^2+y^2)}}\, dx\, dy$$

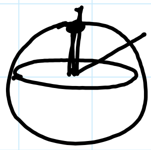
$$\begin{aligned} x &= \rho \cos t \\ y &= \rho \sin t \end{aligned} : \quad \int_{0 \le \rho \le R} \frac{R}{\sqrt{R^2 - \rho^2}}\, \rho\, d\rho\, dt = 2\pi R \int_0^R \frac{\rho}{\sqrt{R^2 - \rho^2}}\, d\rho$$

$y = \rho \sin t$: $\displaystyle\int_{\substack{0\le\rho\le R\\ 0\le t\le 2\pi}} \sqrt{R^2-\rho^2} \quad \int_0 \sqrt{R^2-\rho^2}$

$$= 2\pi R \left[-\sqrt{R^2-\rho^2} \right]_0^R = \boxed{2\pi R^2}$$

$\Rightarrow$ Area della sfera: $4\pi R^2$

ES: Rifare il calcolo utilizzando le coordinate polari-sferiche: $p(\phi,\theta) = (\underbrace{R\sin\phi}\cos\theta, \underbrace{R\sin\phi}\sin\theta, R\cos\phi)$, con $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$. Verificare che



$$\| \underset{\partial_\phi p}{\overset{\shortparallel}{p_\phi}} \times \underset{\partial_\theta p}{\overset{\shortparallel}{p_\theta}} (\phi,\theta) \| = R^2 \sin\phi$$

$$\partial_\phi p(\phi,\theta) = (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi)$$
$$\partial_\theta p(\phi,\theta) = (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0)$$

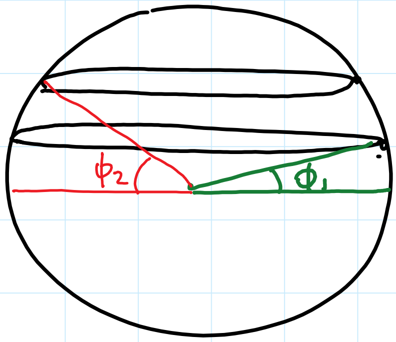
$$\|\partial_\phi p\times\partial_\theta p\|^2 = \left[R^2\cos\phi\sin\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{1})\right]^2 + (R^2\sin^2\phi\cos\theta)^2 + (R^2\sin^2\phi\sin\theta)^2$$
$$= R^4\cos^2\phi\sin^2\phi + R^4\sin^4\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{1})$$
$$= R^4\sin^2\phi(\underbrace{\cos^2\phi+\sin^2\phi}_{1}) = R^4\sin^2\phi \Rightarrow \|\partial_\phi p\times\partial_\theta p\| = R^2\sin\phi$$

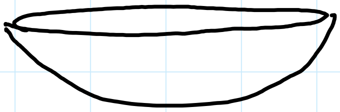
$$\text{Area (Sfera)} = \int_{\substack{0\le\phi\le\pi\\ 0\le\theta\le 2\pi}} \|\partial_\phi p\times\partial_\theta p\|(\phi,\theta)\,d\phi\,d\theta = \int_{[0,\pi]_\phi\times[0,2\pi]_\theta} R^2\sin\phi \, d\phi\, d\theta$$

$$= 2\pi R^2\int_0^\pi \sin\phi\,d\phi = \underline{4\pi R^2}$$

ES. Calcolare area/volume della superficie

ES. Calcolare area/volume della superficie costituita dal settore di una sfera/palla di raggio R e co-latitudine ϕ compresa tra ϕ_1 e ϕ_2, con $0 \le \phi_1 < \phi_2 \le \pi$.





ESERCIZIO. $z = x^2+y^2$, $\overset{D}{x^2+y^2 \le 1}$: calcolare l'area di tale superficie: si tratta di una sup. cartesiana: $(x,y) \mapsto (x,y,x^2+y^2)$

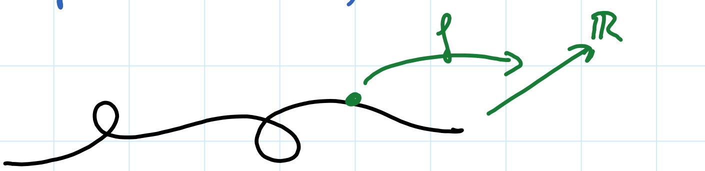
$$\int_D 1 \, \underbrace{\|D_x p \times D_y p\|}_{\sqrt{1+\|\nabla(x^2+y^2)\|^2}}(x,y)\,dx\,dy = \int_{x^2+y^2\le 1} \sqrt{1+4x^2+4y^2}\,dx\,dy$$

$$= 2\pi \int_0^1 \rho\sqrt{1+4\rho^2}\,d\rho = \dots$$

ESERCIZIO. (Interpretazione dell'integrale curvilineo)

Sia $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ curva

$f : \gamma([a,b]) \longrightarrow \mathbb{R}$

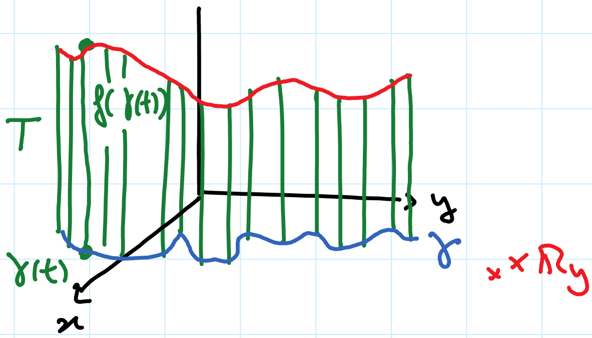


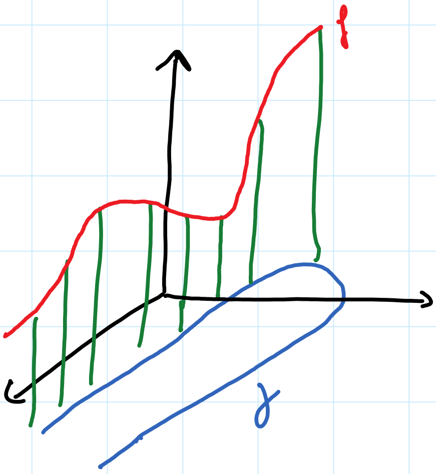
$$\int_\gamma f\,ds = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\,dt$$

Sia $\boxed{f \ge 0}$

Posto $\gamma(t) = (\gamma_1(t), \gamma_2(t))$

consideriamo l'insieme in figura





$$T = \{ (\underbrace{\gamma_1(t), \gamma_2(t)}_{\gamma(t)}, z) : \underbrace{t \in [a,b], \quad 0 \le z \le f(\gamma(t))}_{D} \}$$

Si tratta del sostegno della superficie parametrica

$$p : (t,z) \in D \longmapsto (\gamma_1(t), \gamma_2(t), z)$$

Allora $\boxed{\text{Area}(T) = \int_\gamma f \, ds}$

OSS: $\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$

$$= \int_a^b f(\gamma(t)) \sqrt{\gamma_1'^2(t) + \gamma_2'^2(t)} \, dt$$

$$p(t,z) = (\gamma_1(t), \gamma_2(t), z)$$
$$t \in [a,b], \quad z \in [0, f(\gamma(t))]$$

$$\partial_t p(t,z) = (\gamma_1'(t), \gamma_2'(t), 0)$$
$$\partial_z p(t,z) = (0, 0, 1)$$

$$\|\partial_t p \times \partial_z p\|^2 = 0^2 + \gamma_2'(t)^2 + \gamma_1'^2(t)$$

$$\text{Area}(T) = \int_{\substack{t \in [a,b] \\ 0 \le z \le f(\gamma(t))}} \sqrt{\gamma_1'^2(t) + \gamma_2'^2(t)} \, dt \, dz = \int_a^b \int_0^{f(\gamma(t))} \sqrt{\gamma_1'^2(t) + \gamma_2'^2(t)} \, dz \, dt$$

$$= \int_a^b f(\gamma(t)) \sqrt{\gamma_1'^2(t) + \gamma_2'^2(t)} \, dt = \int_\gamma f \, ds.$$

Commenti vari

INTEGRALI SUPERFICIALI

$p : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ mp. parametrica, $f : p(D) \longrightarrow \mathbb{R}$

Calcolare $\displaystyle\int_p f \, d\sigma_p$ / Area di p.

$$\| $$

$$\int_D f(p(u,v)) \, \|\partial_u p \times \partial_v p\| \, du\,dv$$

Problema tipo

Sia $z = x^2 + 5y^2$ la superficie Σ, con $z \in [1,3]$.
Calcolare l'integrale di $z^2 - 5y$ su Σ.

Traduzione: superficie cartesiana $(x,y) \longmapsto \boxed{(x, y, x^2+5y^2)}$

? Dove variano x e y ?

$$z \in [1,3] \iff \underbrace{1 \le x^2 + 5y^2 \le 3}_{D}$$

$f(x,y,z) = z^2 - 5y$

$$\int_p f \, d\sigma_p = \int_D f(x, y, x^2+5y^2) \sqrt{1 + \|\nabla(x^2+5y^2)\|^2} \, dx\,dy$$

$$= \int_D (z^2(x,y) - 5y) \sqrt{\dots} \, dx\,dy$$

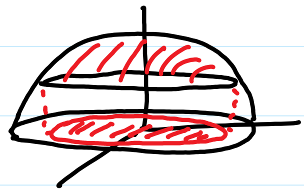
$$= \int_D ((x^2+5y^2)^2 - 5y) \sqrt{\dots} \, dx\,dy .$$

ES. $\Sigma : x^2 + y^2 + z^2 = 4$ con $\underline{z \in [1,2]}$. Integrare x^2 sulla superficie...

ES. $\Sigma: x^2+y^2+z^2=4$ con $z \in [1,2]$. Integrare x^2 sulla superficie.

$$z = \pm\sqrt{4-x^2-y^2}$$

$$z = \sqrt{4-x^2-y^2}$$



$$1 \le \sqrt{4-x^2-y^2} \le 2$$

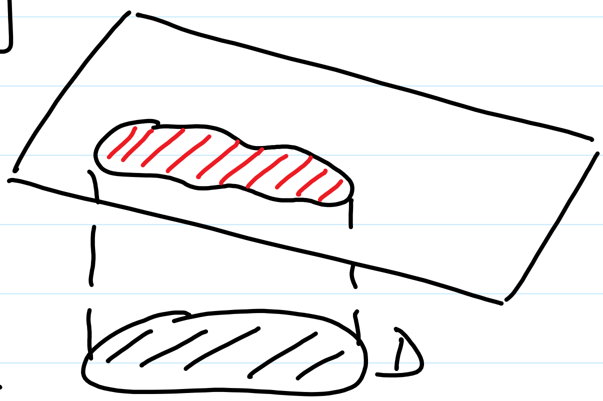
$$1 \le 4-x^2-y^2 \le 4 \iff \underbrace{x^2+y^2 \le 3}_{D}$$

ES. Area. $\boxed{z = 1-4x-3y}$

El.to d'area della sup. cartesiana è

$$\sqrt{\|\nabla(1-4x-3y)\|^2+1}$$

$$= \sqrt{1+4^2+3^2} = \sqrt{26}$$



Area è $\displaystyle\int_D \sqrt{26}\, dx\, dy = \sqrt{26} \times \text{Area}(D)$

Se $D = \{(x,y) : \frac{x^2}{4} + y^2 \le 1\}$

$$\left(\frac{x}{2}\right)^2 + y^2 \le 1 \qquad \pi \times 2 \times 1 = 2\pi$$

L'area della superficie è $\sqrt{26} \times 2\pi$.

ES $\Sigma: z = x^2+y^2$ con $\boxed{\begin{cases} x^2+y^2-y \le 0 \\ x \ge 0 \end{cases} \quad D}$

Integrare $\dfrac{x}{\sqrt{4z+1}}$ su Σ.

① Sup. cartesiana $z = x^2+y^2 \rightsquigarrow$ El.to d'area

$$\sqrt{\|\nabla(x^2+y^2)\|^2+1}$$

$$= \sqrt{1+4x^2+4y^2}$$

② $$\int_D \frac{x}{\cancel{\sqrt{1+4(x^2+y^2)}}} \cancel{\sqrt{1+4(x^2+y^2)}}\, dx\, dy = \int_D x\, dx\, dy$$

$$\underbrace{x^2-5x} + \underbrace{4y^2+8y} \leq 10$$

$$X^2+aX = \left(X+\frac{a}{2}\right)^2 - \frac{a^2}{4}$$

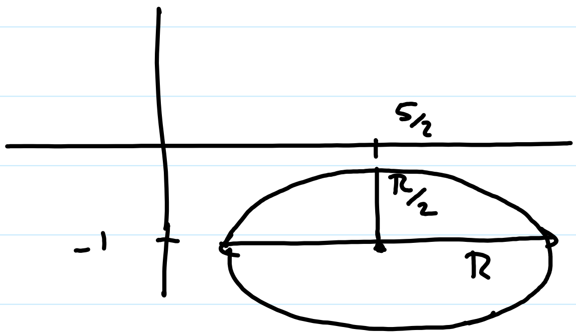
$$\left(x-\frac{5}{2}\right)^2 - \frac{25}{4} + 4(y^2+2y) \leq 10$$

$$\left(x-\frac{5}{2}\right)^2 - \frac{25}{4} + 4\left((y+1)^2-1\right) \leq 10$$

$$\left(x-\frac{5}{2}\right)^2 + 4(y+1)^2 \leq \underbrace{10+\frac{25}{4}+4}_{R^2}$$

$$\left(x-\frac{5}{2}\right)^2 + \left(\frac{y+1}{\frac{1}{2}}\right)^2 \leq R^2$$

$$\left(\frac{x-\frac{5}{2}}{R}\right)^2 + \left(\frac{y+1}{\frac{R}{2}}\right)^2 \leq 1$$



$$\left(\frac{x-a}{A}\right)^2 + \left(\frac{y-b}{B}\right)^2 \leq 1$$

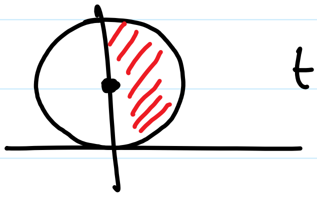
$$\begin{cases} \dfrac{x-a}{A} = \rho\cos t \\ \dfrac{y-b}{B} = \rho\sin t \end{cases} \qquad \begin{cases} x = a + A\rho\cos t \\ y = b + B\rho\sin t \end{cases} \qquad \begin{matrix} t\in[0,2\pi] \\ 0<\rho\leq 1 \end{matrix}$$

Esempio precedente: $\begin{cases} x^2+y^2-y \le 0 \\ x \ge 0. \end{cases}$

$x^2+(y-\frac{1}{2})^2-\frac{1}{4} \le 0 \qquad x^2+(y-\frac{1}{2})^2 \le \frac{1}{4}$:

disco centro $(0,\frac{1}{2})$ raggio $\frac{1}{2}$

$\begin{cases} x = \rho\cos t \\ y = \frac{1}{2}+\rho\sin t \end{cases} \qquad \begin{array}{l} 0 \le \rho \le \frac{1}{2} \\ t \in [0, 2\pi] \end{array} \qquad \Bigg| \qquad \begin{cases} x = \frac{\rho}{2}\cos t \\ y = \frac{1}{2}+\frac{\rho}{2}\sin t \end{cases} \qquad \begin{array}{l} 0 \le \rho \le 1 \\ t \in [0, 2\pi] \end{array}$



$t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

ES. "La curva del piano $y = 3x^2-5$." $x \in [0,1]$

Filo metallico di densità $\mu(x,y) = xy$

Trovare la massa del filo.

Curva: $\gamma(x) = (x, 3x^2-5)$

$$\int_\gamma \mu \, ds = \int_0^1 x \underbrace{y(x)}_{3x^2-5} \sqrt{1+((3x^2-5)')^2}\, dx$$