Padova, November 18Th

Steady states and stability

Definition (Equilibrium)

Given a $n-$dimensional first order system of differential equations

$$\dot{\mathbf{y}} = F(\mathbf{y})$$

We say that $\mathbf{y}^*$ is an **asymptotically stable equilibrium** if every solution $\mathbf{y}(t)$ which starts near $\mathbf{y}^*$ converges to $\mathbf{y}^*$ as $t \to \infty$.

SS **steady state** $\equiv$ **stationary solution** $\equiv$ **rest point**

Stable equilibria

Definition ((locally) asymptotically stable)

A steady state solution $\mathbf{y}^*$ of the system $\dot{\mathbf{y}} = F(\mathbf{y})$ is called **(locally) asymptotically stable** if any solution $\mathbf{y}(t)$ which starts near $\mathbf{y}^*$ converges to $\mathbf{y}^*$ as $t \to \infty$.

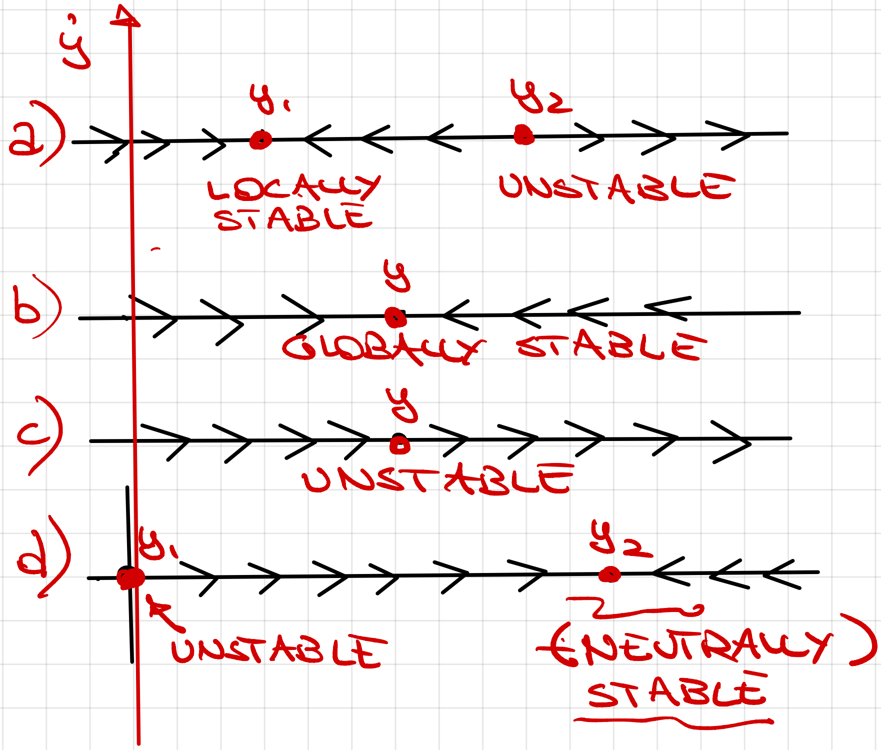
Definition (globally asymptotically stable)

A steady state solution $\mathbf{y}^*$ of the system $\dot{\mathbf{y}} = F(\mathbf{y})$ is called **globally asymptotically stable** if just about every solution of $\dot{\mathbf{y}} = F(\mathbf{y})$ tends to $\mathbf{y}^*$ as $t \to \infty$.

$\dot{y} = -y$

Definition (neutrally stable)

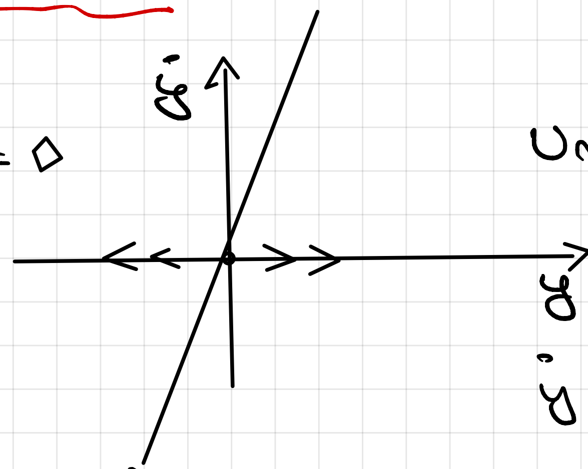
A steady state solution $\mathbf{y}^*$ of the system $\dot{\mathbf{y}} = F(\mathbf{y})$ is called **neutrally stable** if it is not locally asymptotically stable and if all solutions which start close enough to $\mathbf{y}^*$ stay close to $\mathbf{y}^*$ as $t \to \infty$.



i) $y' = 3y$ SS $y^* = 0$

$f(y)$

$f'(y) = 3$



Unstable

$f'(y^*) > 0$

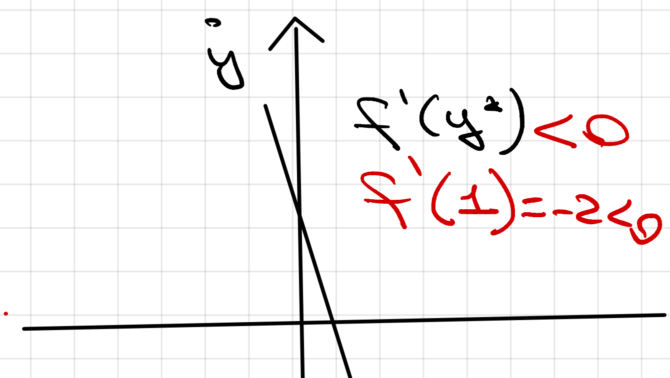
$f'(0) = 3 > 0$

ii) $y' = \underline{-2(y-1)}$ SS $y^* = 1$

$f(y)$

$f(y) = 0$

$\boxed{f'(y) = -2}$

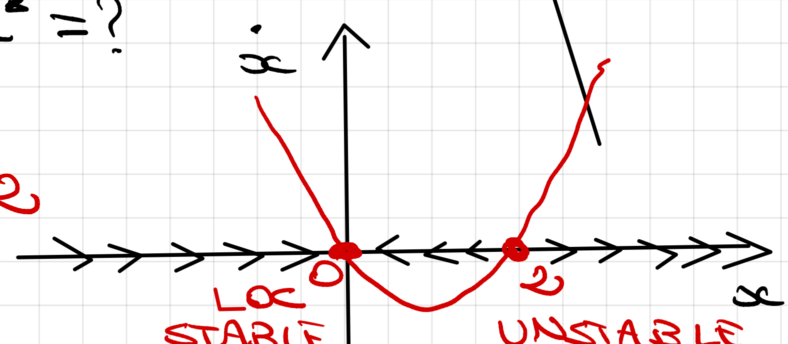


$f'(y^*) < 0$

$f'(1) = -2 < 0$

iii) $x' = \underbrace{x(x-2)}_{f(x)}$ SS $x^* = ?$

$f(x) = 0 \Rightarrow x_1^* = 0, \; x_2^* = 2$



$f'(x) = 2x - 2$

$f'(x_1^*) = f'(0) = -2 < 0$ LOC. STAB

$f'(x_2^*) = f'(2) = 2 > 0$ UNSTABLE

Stability of Linear Systems of ODE

Given the Linear System

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$$

The general solution is

λ_i : eigenvalues

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{w}_1 + c_2 e^{\lambda_2 t}\mathbf{w}_2 + \ldots + c_n e^{\lambda_n t}\mathbf{w}_n$$

The constant solution $\mathbf{x} = \mathbf{0}$ is always a steady state (SS) of the linear system of ODES $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$

Let $t \to +\infty$

STABILITY

$\lim_{t \to +\infty} e^{\lambda_i t} = 0$ iff $\lambda_i < 0$

Stability of Linear Systems of ODE

Theorem

- *If every real eigenvalue of A is negative (every complex eigenvalue of A has negative real part), then* $\mathbf{x} = \mathbf{0}$ *is a* **globally asymptotic stable** *SS: every solution tends to* $\mathbf{0}$ *as* $t \to \infty$
- *If A has* at least one *a positive real eigenvalue (or a complex eigenvalue with positive real part), then* $\mathbf{x} = \mathbf{0}$ *is an* **unstable** *SS: just about every solution moves away from the origin as* $t \to \infty$
- *If A has a zero eigenvalue (or a purely imaginary eigenvalue) that does not have a complete set of independent eigenvectors, then* $\mathbf{x} = \mathbf{0}$ *is an* **unstable** *SS: just about every solution moves away from the origin as* $t \to \infty$
- *If A has a zero eigenvalue (or a purely imaginary eigenvalue), if all such eigenvalues have a complete set of independent eigenvectors, and if all the other eigenvalues are negative (or negative part), then* $\mathbf{x} = \mathbf{0}$ *is a* **neutrally stable** *SS.*

Local Stability of NonLinear differential equation

Given the **autonomous** differential equation

$$\dot{y}(t) = f(y)$$

Definition ((locally) asymptotically stable)

A steady state solution $\mathbf{y}^*$ is called **(locally) asymptotically stable** if any solution $f(\mathbf{y}^*) = 0$ and $f'(\mathbf{y}^*) < 0$.

Definition (unstable)

A steady state solution $\mathbf{y}^*$ is called **unstable** if any solution $f(\mathbf{y}^*) = 0$ and $f'(\mathbf{y}^*) > 0$.

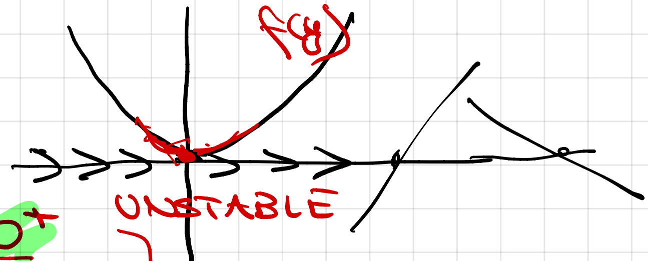
$$y' = \underbrace{3e^y - 3y - 3}_{f(y)}$$

$y^* = 0$ check it is an eq.

$f(y^*) = f(0) = 3 \cdot e^0 - 3 \cdot 0 - 3 = 0$ ✓ SS

$f'(y) = 3e^y - 3$

$f'(0) = 3 - 3 = 0$



$y_1 = 0 - \varepsilon \qquad \lim_{y \to 0^-} f(y) = 0^+$

$y_2 = 0 + \varepsilon \qquad \lim_{y \to 0^+} f(y) = 0^+$

Local Stability of NonLinear Systems of ODE

Theorem (Sufficient conditions)

Given the **autonomous** *NonLinear System*

$$\dot{x} = x^2 y + 3 = F_1 \quad \dot{y} = \ln y + x = F_2$$

$$\dot{\mathbf{y}}(t) = F(\mathbf{y}) = \begin{bmatrix} x^2 y + 3 \\ \ln y + x \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

- *If each eigenvalue of the jacobian matrix* $DF(\mathbf{y}^*)$ *of* F *is negative (or has negative real part), then* $\mathbf{y}^*$ *is a (local) asymptotic stable SS.*
- *If* $DF(\mathbf{y}^*)$ *has at least one positive real eigenvalue (or one with positive real part), then* $\mathbf{y}^*$ *is an unstable SS.*

Example

Given the following System of ODEs

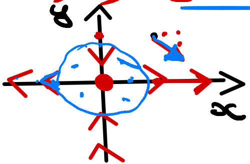
$$\begin{cases} \dot{x} = \underbrace{2x}_{F_1} = 0 \\ \dot{y} = \underbrace{-2y}_{F_2} = 0 \end{cases} \quad \textit{Steady state}(0,0).$$

The jacobian is

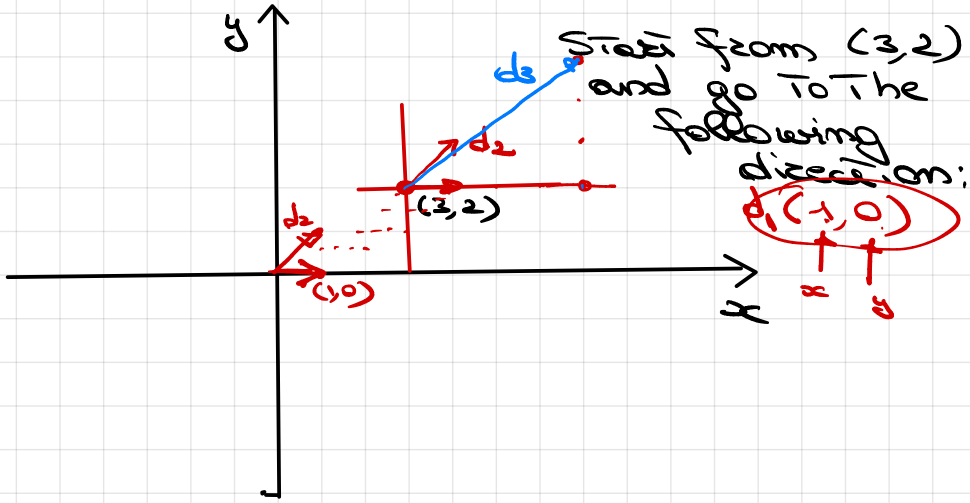
$$\begin{pmatrix} \nabla F_1 \\ \nabla F_2 \end{pmatrix} = DF = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

eigenvalues $\lambda_i = 2, -2$ $\quad \exists\, \lambda_i = 2 > 0$

$\Rightarrow (0,0)$ is UNSTABLE



(x,y)	$(\dot{x},\dot{y})$ vector
$(0,0)$	$(0,0)$
$(1,0)$	$(2,0)$
$(0,1)$	$(0,-2)$
$(1,1)$	$(2,-2)$ $//(1,-1)$
$(-1,0)$	$(-2,0)$



y
x
Start from (3,2)
and go To The
following
direction:
d_1(1,0)
x
y
d_3
d_2
(3,2)
d_2
(1,0)

$d_2 = (1, 1)$
$d_3 = (4, 3)$

Example

Given the following System of ODEs

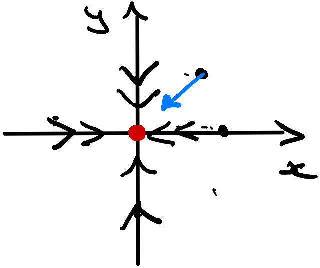
$$\begin{cases} \dot{x} = -2x = 0 \\ \dot{y} = -2y = 0 \end{cases} \quad \textit{Steady state}(0,0).$$

The jacobian is

$$DF = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = -2 < 0$$

Theo.

(0,0). STABLE



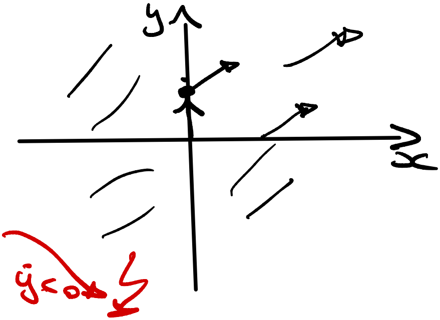
(x,y)	$(\dot{x},\dot{y})$
(1,0)	(−2,0)
(0,1)	(0,−2)
(0,−1)	(0,2)
(1,1)	(−2,−2)

Example

Given the following System of ODEs

$$\begin{cases} \dot{x} = x^2 + y^2 = 0 \\ \dot{y} = 1 > 0 \neq 0 \end{cases}$$

$\nexists$ *NO Steady states* $\longrightarrow$ $\dot{x} > 0$, $\dot{y} > 0$



(x, y)	$(\dot{x}, \dot{y})$
$(0,0)$	$(0,1)$
$(0,y)$	$(y^2,1)$
$(0,1)$	$(1,1)$
$(1,0)$	$(1,1)$

Example

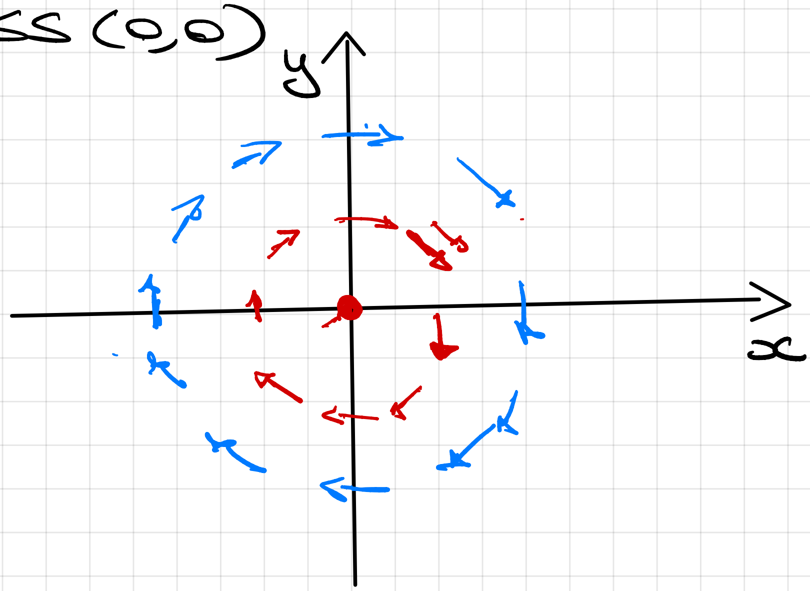
$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$DF \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

complex

SS (0,0)



(x, y)	$(\dot{x}, \dot{y})$
(0, 0)	(0, 0)
(1, 0)	(0, −1)
(0, 1)	(1, 0)
(1, 1)	(1, −1)
(2, 0)	(0, −2)
(−1, −1)	(−1, 1)

Example

Given the following System of ODEs

$$\begin{cases} \dot{x} = x(4 - x - y) = 0 \\ \dot{y} = y(6 - y - 3x) = 0 \end{cases}$$

The jacobian is

$$DF = \begin{pmatrix} 4 - 2x - y & -x \\ -3y & 6 - 2y - 3x \end{pmatrix}$$

SS O(0,0)
A(4,0)
B(1,3)
C(0,6)

$$DF\Big|_{(0,0)} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \qquad \lambda_{1,2} = 4, 6 \underline{> 0}$$

(0,0) UNSTABLE

$$DF\Big|_{(4,0)} = \begin{pmatrix} -4 & -4 \\ 0 & -6 \end{pmatrix} \qquad \begin{array}{l} \lambda_1 = -4 \\ \lambda_2 = -6 \end{array}$$

(4,0) STABLE

$$DF\Big|_{(1,3)} = \begin{pmatrix} -1 & -1 \\ -9 & -3 \end{pmatrix}$$

$$\lambda^2 + 4\lambda - 6 = 0$$

$$\lambda = -2 \pm \underbrace{\sqrt{4+6}}$$

$\lambda_1 < 0$

$\lambda_2 > 0$

(1,3) UNSTAB

$$DF\Big|_{(0,6)} = \begin{pmatrix} -2 & 0 \\ -18 & -6 \end{pmatrix} \qquad \begin{array}{l} \lambda_1 = -2 < 0 \\ \lambda_2 = -6 < 0 \end{array}$$

(0,6) Stable

Vector fields

For any point of the plane (x, y) draw the vectors $(\dot{x}, \dot{y})$

$$\begin{cases} \dot{x} = x(4 - x - y) \\ \dot{y} = y(6 - y - 3x) \end{cases}$$

$$\begin{array}{ccc} (x, y) & | & (\dot{x}, \dot{y}) \\ (1,0) & \to & (3,0) \\ (0,1) & \to & (0,5) \\ (1,1) & \to & (2,2) \\ (3,3) & \to & (-6,-18) \end{array}$$

Phase portraits - phase diagram