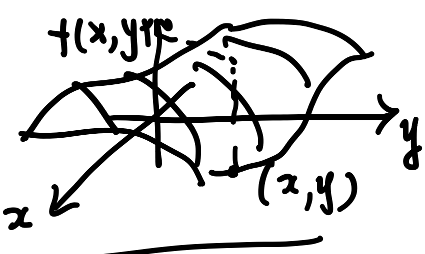
Tuesday, 7 February 2023 12:54



Functions of Several vars:

$f(x,y) \in \mathbb{R}$

$f(x,y,z) \in \mathbb{R}$

$f(x,y) := x+y \in \mathbb{R}$

$:= xy \in \mathbb{R}$

$:= x e^{y \sin x} \in \mathbb{R}$

$f(x,y,z) := x^2+y^2+z^2 \in \mathbb{R}$

$:= z\sin(xy) \in \mathbb{R}$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$\mathbb{R}^3$

$f = f(x_1, \dots, x_d) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$
(D: domain)

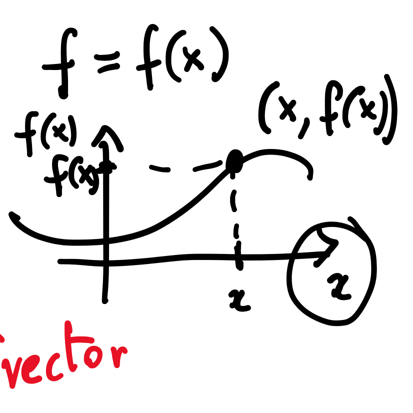
(numerical funct of d - vars)
vectors

$F(x,y) := (xy, x+y)$ ←

$F(1,1) = (1,2)$

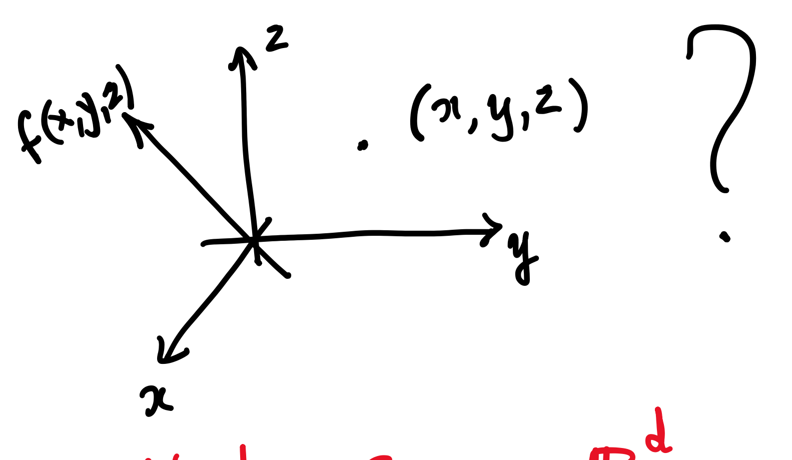
$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$F: D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$



$F(x,y) := \left(x e^y, \log(xy), \frac{x}{y}\right)$ ← vector
(vector)

$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

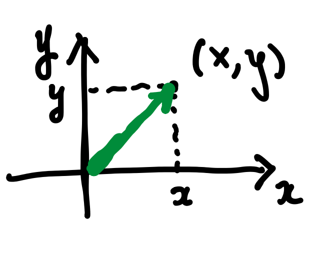


Vector Space $\mathbb{R}^d$

Def: $\mathbb{R}^d := \{ \vec{x} = (x_1, x_2, \dots, x_d) : x_j \in \mathbb{R},\ j = 1 \dots d\}$
(vector)

On $\mathbb{R}^d$ it is defd

$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

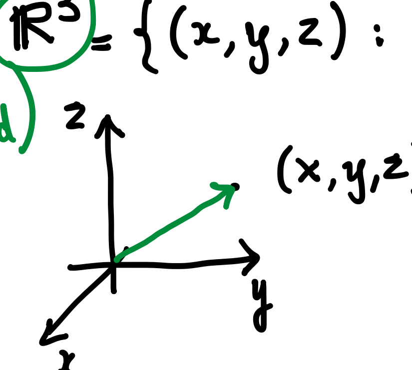


- a sum

$\vec{x}$
$(x_1, \dots, x_d) + (y_1 \dots y_d)$

$:= (x_1+y_1, x_2+y_2, \dots, x_d+y_d)$

$\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$



- a product by scalars

$\lambda (x_1 \dots, x_d) :=$ (by def) $(\lambda x_1, \lambda x_2, \dots, \lambda x_d)$
$\lambda \in \mathbb{R}$

Ex $\mathbb{R}^2$

- $(1,2)+(-3,5) = (1-3, 2+5)$ [$1+(-3)$]
$= (-2, 7)$
- $3(1,2) = (3,6)$.

With these two operations $\mathbb{R}^d$ becomes a vector space, in the sense that the following props. hold:

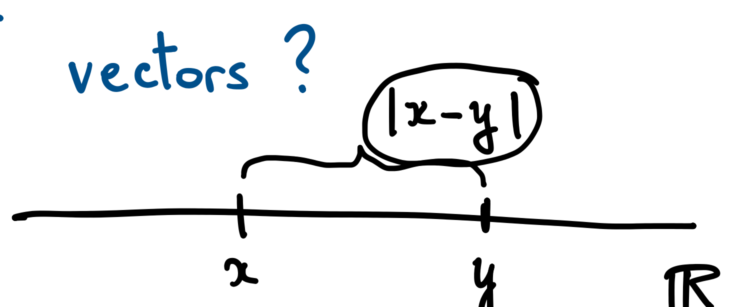
- (commutativity sum) $\forall\, \vec{x}, \vec{y} \in \mathbb{R}^d \Rightarrow \vec{x}+\vec{y} = \vec{y}+\vec{x}$
- (associativity sum) $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$
$\forall\, \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^d$
- (zero) $\exists\, \vec{0} = (0,0,\dots,0) :$ $\vec{x}+\vec{0} = \vec{x}$ $\forall\, \vec{x} \in \mathbb{R}^d$
(: such that)
- (opposite) $\forall\, \vec{x} \in \mathbb{R}^d\ \exists\, \vec{y} \in \mathbb{R}^d : \vec{x}+\vec{y} = \vec{0}$
(if $\vec{x} = (x_1, \dots x_d) \Rightarrow \vec{y} = (-x_1, -x_2, \dots, -x_d)$)
- (assoc. for prod by scalars)
$\mu(\lambda \vec{x}) = (\mu\lambda)\vec{x}$ $\quad \mu, \lambda \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^d$
($\mu\lambda \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^d$)
- (unity)
$1\vec{x} = \vec{x}$ $\quad \forall\, \vec{x} \in \mathbb{R}^d$
- (distributivity props)
$\lambda(\vec{x}+\vec{y}) = \lambda\vec{x} + \lambda\vec{y}$ $\quad \forall\, \lambda \in \mathbb{R},\ \forall\, \vec{x}, \vec{y} \in \mathbb{R}^d$
$(\lambda+\mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$ $\quad \forall\, \lambda, \mu \in \mathbb{R},\ \forall\, \vec{x} \in \mathbb{R}^d$

Our first goal is to give a definition of limit for $f = f(\vec{x}) = f(x_1, x_2, \dots, x_d)$

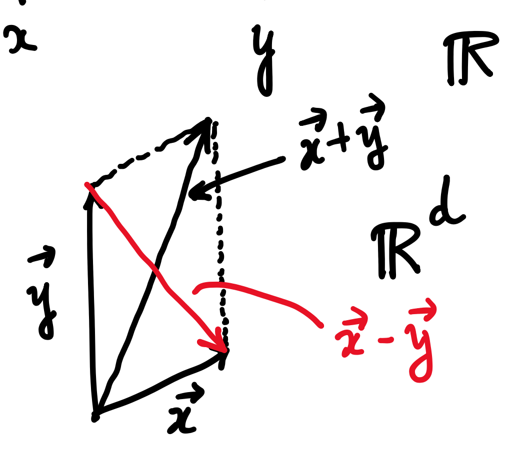
$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = \ell \quad (*)$$

Intuitive idea: $f(\vec{x})$ is close to ℓ when $\vec{x}$ is close to $\vec{x}_0$

Pb: How do we measure distance between vectors?

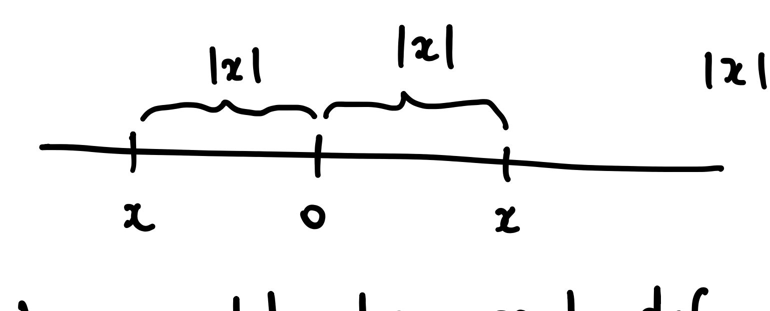


$\mathbb{R} = \mathbb{R}^1$



$\mathbb{R}^d$

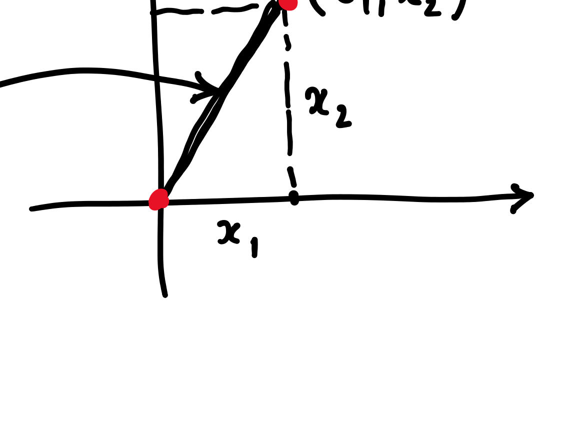
$\vec{x} = (x_1 \text{—} x_d)$
$\vec{y} = (y_1 \text{—} y_d)$ $\Rightarrow$ $\vec{x}-\vec{y} = (x_1-y_1, \dots x_d-y_d)$



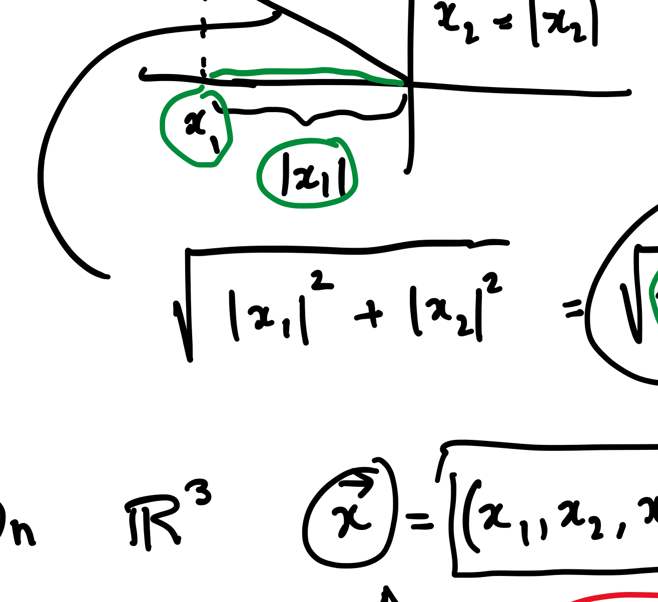
$|x|$ = distance of x to 0

What could be a good def of dist. between $\vec{x}$ and $\vec{0}$?

On $\mathbb{R}^2$ $\vec{x} = (x_1, x_2)$

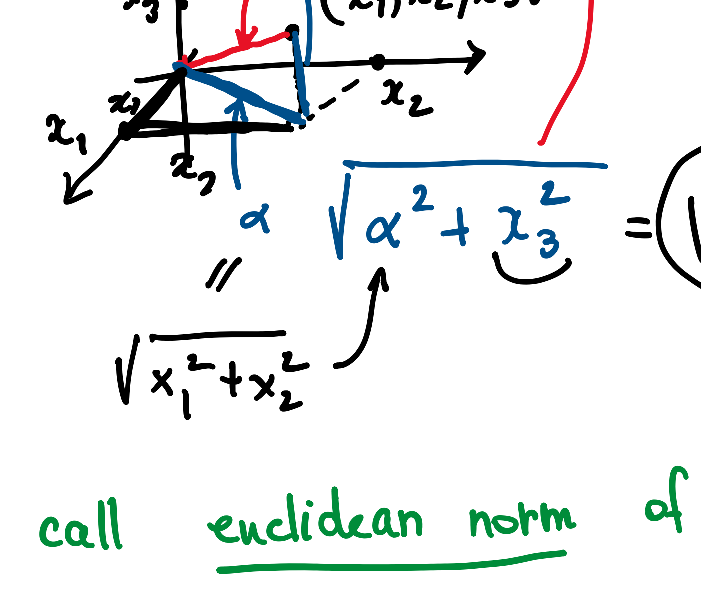


$\sqrt{x_1^2 + x_2^2}$



$\sqrt{|x_1|^2 + |x_2|^2} = \sqrt{x_1^2 + x_2^2}$ is the dist between $\vec{x}$ and $\vec{0}$

On $\mathbb{R}^3$ $\vec{x} = (x_1, x_2, x_3)$



$\alpha = \sqrt{x_1^2 + x_2^2}$, $\sqrt{\alpha^2 + x_3^2} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Def: We call euclidean norm of $\vec{x} = (x_1, \dots x_d)$ the quantity

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_d^2} = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}.$$

$= 0 \Leftrightarrow x_1^2 + \dots + x_d^2 = 0$
$\Leftrightarrow x_i^2 = 0 \quad \forall i$
$\Leftrightarrow x_i = 0 \quad \forall i$

Rmk: $\|\vec{x}\|$ is always defd.

Properties of $\|\cdot\|$:

1) (positivity) $\|\vec{x}\| \geq 0 \quad \forall\, \vec{x} \in \mathbb{R}^d$
2) (vanishing) $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$
3) (homogeneity)
$\|\lambda \vec{x}\| = |\lambda| \|\vec{x}\|$ $\quad \lambda \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^d$
4) (triang. inequality)
$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
$\forall\, \vec{x}, \vec{y} \in \mathbb{R}^d$

Props of $|\cdot|$

$|x| \geq 0 \quad \forall\, x \in \mathbb{R}$

$|x| = 0 \Rightarrow x = 0$

$|xy| = |x| \cdot |y|$

$|x+y| \leq |x| + |y|$

$|1+(-1)| \quad |1| + |-1|$
$|0| = 0 < 1 + 1 = 2$