Thursday, 9 February 2023 12:55

Limit for a sequence of vectors

Recall: If $(x_n) \subset \mathbb{R}$, $x_0, x_1, x_2, \ldots$

we say that $x_n \to \ell \in \mathbb{R}$ $n \to +\infty$

if " dist. between x_n and ℓ becomes small provided n gets big enough"

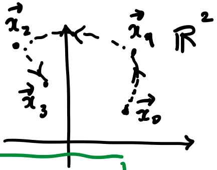
Formally:

$$\boxed{\forall \varepsilon > 0 \quad \exists N : \quad |x_n - \ell| \leq \varepsilon, \quad \forall n \geq N.}$$ ←

Def: Let $(\vec{x}_n) \subset \mathbb{R}^d$,

We say that $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$

when $n \to +\infty$ if



$$\boxed{\forall \varepsilon > 0 \quad \exists N : 0 \leq \underbrace{\|\vec{x}_n - \vec{\ell}\,\|}_{} \leq \varepsilon, \quad \forall n \geq N.}$$ ← (*)

Rmk: $(\|\vec{x}_n - \vec{\ell}\,\|)$ is a seq of pos. numbers

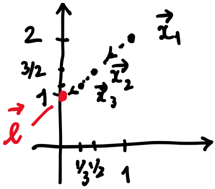
and (*) $\Leftrightarrow$

$$\lim \|\vec{x}_n - \vec{\ell}\,\| = 0.$$

$$\vec{x}_n \xrightarrow{\;n\to+\infty\;} \underset{\uparrow}{\vec{\ell}} \iff \boxed{\|\vec{x}_n - \vec{\ell}\,\| \longrightarrow 0}$$

(as vectors) (as numbers)

Ex 1.1.5 Show that $\underbrace{\overset{\vec{x}_n}{\left(\frac{1}{n}, \frac{n+1}{n}\right)}} \longrightarrow \underbrace{\overset{\vec{\ell}}{(0,1)}}$



n	$\left(\frac{1}{n}, \frac{n+1}{n}\right)$
1	$(1,2)$
2	$(1/2, 3/2)$
3	$(1/3, 4/3)$
$\vdots$	
big n	$\left(\underset{\underset{0}{\approx}}{\frac{1}{n}}, \underset{\underset{1}{\approx}}{\frac{n+1}{n}}\right) \sim (0,1)$

To prove that $\underset{\vec{x}_n}{\left(\frac{1}{n}, \frac{n+1}{n}\right)} \longrightarrow \underset{\vec{\ell}}{(0,1)}$

we have to prove that

$$= \left\| \overset{\vec{x}_n}{\left(\frac{1}{n}, \frac{n+1}{n}\right)} - \overset{\vec{\ell}}{(0,1)} \right\| \xrightarrow[n\to+\infty]{} 0$$

$$\|$$

$$\left(\frac{1}{n} - 0, \frac{n+1}{n} - 1\right) = \left(\frac{1}{n}, \frac{1}{n}\right)$$

$\cancel{1} + 1/n - \cancel{1}$

$$= \left\| \left(\frac{1}{n}, \frac{1}{n} \right) \right\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2}$$

$$= \sqrt{2 \cdot \frac{1}{n^2}}$$

$$= \boxed{\frac{\sqrt{2}}{n}}$$

$\sqrt{n^2} = n$

$\underset{0}{\underset{\vee}{n}} \in \mathbb{N}$

$$\Rightarrow \quad \| \vec{x}_n - \vec{\ell} \| = \boxed{\frac{\sqrt{2}}{n}} \xrightarrow[n \to +\infty]{} 0. \quad \blacksquare$$

Ex 1.7.2 (do all questions)

iii) $\vec{x}_n := \left(\frac{1}{n}, \frac{1}{n^2}, \sin \frac{1}{n} \right)$

Discuss if $(\vec{x}_n)$ has a limit and, if yes compute it.

Here $$\vec{x}_n = \boxed{\left(\boxed{\frac{1}{n}}, \frac{1}{n^2}, \sin \frac{1}{n} \right)} \xrightarrow{?} \underset{\vec{0}}{\underset{\shortparallel}{\boxed{(0,0,0)}}}$$

$n \to +\infty \qquad \downarrow 0 \qquad \downarrow 0 \qquad \downarrow 0$

To check this we have to check if

$$\boxed{\|\vec{x}_n - \vec{\ell}\,\| \longrightarrow 0} \iff$$

$$\boxed{\Big\| \underbrace{\left(\tfrac{1}{n}, \tfrac{1}{n^2}, \sin\tfrac{1}{n}\right) - (0,0,0)} \Big\|}$$

$$\left(\frac{1}{n}-0, \frac{1}{n^2}-0, \sin\frac{1}{n}-0\right) = \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right)$$

$$= \left\| \left(\textcircled{\tfrac{1}{n}}, \textcircled{\tfrac{1}{n^2}}, \sin\frac{1}{n} \right) \right\|$$

$$= \underbrace{\sqrt{\Big(\underbrace{\frac{1}{n^2}}_{\downarrow\, 0} + \underbrace{\frac{1}{n^4}}_{\downarrow\, 0} + \underbrace{\sin^2\frac{1}{n}}_{\downarrow\, 0} \Big)}}_{\to\, 0} \;\boxed{\longrightarrow \sqrt{0} = 0}$$

∎

Rmk: $(\vec{x}_n) \subset \mathbb{R}^{\textcircled{d}}$

$\vec{x}_1 = (x_{1,1}, x_{1,2}, \dots, x_{1,d})$

$\vec{x}_2 = (x_{2,1}, x_{2,2}, \dots, x_{2,d})$

$\vec{x}_3 = \vdots$

$\vdots$

$\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$

$\vdots$

Ex : $\vec{x}_n = \Big(\underbrace{\overset{x_{n,1} \approx 0}{\sin 1/n}}_{\searrow 0} , \underset{\searrow}{\overset{x_{n,2} \approx 1}{\cos 1/n}}\Big) \longrightarrow (0,1)$

$x_{n,1} = \sin 1/n \longrightarrow 0$

$x_{n,2} = \cos 1/n \longrightarrow 1$

Prop: If $(\vec{x}_n) \subset \mathbb{R}^d$, $\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$

then:

$$\boxed{\vec{x}_n \longrightarrow \vec{\ell}} = (\ell_1, \ell_2 \dots, \ell_d) \in \mathbb{R}^d$$

$$\Updownarrow \qquad \Uparrow \leftarrow$$

$$\left[\begin{array}{l} x_{n,1} \longrightarrow \ell_1 \in \mathbb{R} \\ x_{n,2} \longrightarrow \ell_2 \in \mathbb{R} \\ \vdots \\ x_{n,d} \longrightarrow \ell_d \in \mathbb{R} \end{array}\right. \qquad n \to +\infty$$

Ex. $\vec{x}_n = \left(\frac{1}{n}, \frac{n+1}{n}\right).$

$x_{n,1} \qquad x_{n,2}$

Since

$$x_{n,1} = 1/n \longrightarrow 0$$

$$x_{n,2} = \frac{n+1}{n} = 1 + 1/n \longrightarrow 1$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow (0,1). \qquad ■$$

Ex. Discuss limit of $\vec{x}_n = \left(\frac{n^2+1}{n^2-1}, \frac{1}{n+1}, \frac{\log n}{n} \right)$

Sol: Here $(\vec{x}_n) \subset \mathbb{R}^3$

$$x_{n,1} = \frac{n^2+1}{n^2-1} \xrightarrow[n \to +\infty]{} \boxed{1} \leftarrow$$

$$x_{n,2} = \frac{1}{n+1} \xrightarrow[n \to +\infty]{} \boxed{0} \leftarrow$$

$$x_{n,3} = \frac{\log n}{n} \xrightarrow[n \to +\infty]{} \boxed{0} \swarrow \quad n \gg_{+\infty} \log n$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow (1,0,0). \qquad ■$$

Rmk: Prev prop says that

$$\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$$

then

$$\vec{x}_n \longrightarrow \underbrace{\vec{\ell}}_{} \in \mathbb{R}^d \Longleftrightarrow$$

$$\left.\begin{array}{l} x_{n,1} \longrightarrow \ell_1 \\ x_{n,2} \longrightarrow \ell_2 \\ \vdots \\ x_{n,d} \longrightarrow \ell_d \end{array}\right\} \in \mathbb{R}.$$

What if one or more components $x_{n,k}$ (k fixed) does not have any lim?

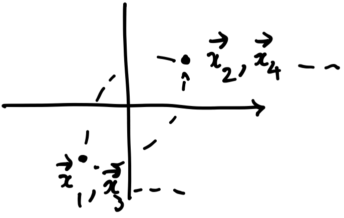
$\Rightarrow$ $(\vec{x}_n)$ cannot have a finite lim!

Ex. $+1, -1, +1, -1, \ldots = (-1)^n$

(example of seq of real numbers without limit)

Ex: $\vec{x}_n = \left((-1)^n, (-1)^n\right)$

n	$\vec{x}_n$
1	$(-1, -1)$
2	$(+1, +1)$
3	$(-1, -1)$



This $\vec{x}_n$ cannot have any finite limit because

$x_{n,1} = (-1)^n \not\longrightarrow$

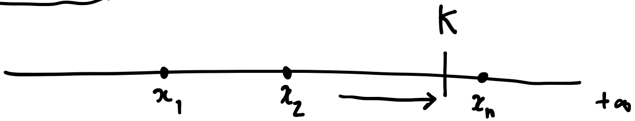
Do Ex 1.7.2 i), v), vi)

Ex 1.7.3 (theoretical ex.).

Infinite Limit (for a seq of vectors)

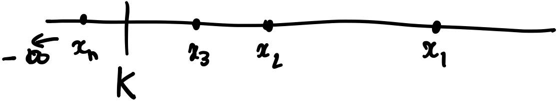
If $(x_n) \subset \mathbb{R}$, we say that

$\boxed{x_n \longrightarrow +\infty}$ if $\forall K \ \exists N : \underline{x_n \geq K} \ \ \forall \underline{n \geq N}$

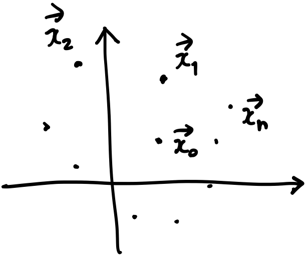


$\boxed{x_n \longrightarrow -\infty}$

$|x_n| \longrightarrow +\infty$



$\forall K \ \exists N : x_n \leq K, \ \forall n \geq N.$



$\|\vec{x}_n\| \longrightarrow +\infty$

Def: Let $(\vec{x}_n) \subset \mathbb{R}^d$. We say that

$$\boxed{\vec{x}_n \longrightarrow \infty_d \quad \text{if} \quad \|\vec{x}_n\| \longrightarrow +\infty}$$

Ex. 1.1.9

$$\vec{x}_n = \left(n, \frac{1}{n}\right)$$

What about $\lim_{n \to +\infty} \vec{x}_n$?

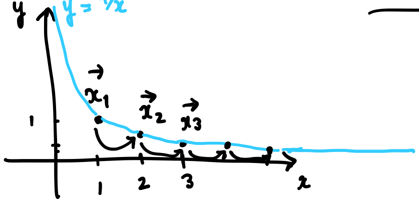
Sol: Here

$$x_{n,1} = n \longrightarrow +\infty \notin \mathbb{R}$$

$$x_{n,2} = \frac{1}{n}$$

$\Rightarrow$ $(\vec{x}_n)$ has not finite lim.

But $\boxed{\vec{x}_n \longrightarrow \infty_2}$



$xy = 1$

$x \quad y$

n	$\vec{x}_n = (n, \frac{1}{n})$
1	$(1, 1)$
2	$(2, \frac{1}{2})$
3	$(3, \frac{1}{3})$
4	$(4, \frac{1}{4})$

Proof: $\|(\vec{x}_n)\| = \sqrt{\underbrace{n^2 + \frac{1}{n^2}}} \longrightarrow +\infty$

$$\xrightarrow[n\to+\infty]{} +\infty$$

Ex : $\vec{x}_n = (n\cos n, n\sin n)$. lim ?

Sol: Here

$$x_{n,1} = \overset{\downarrow}{n}\,\boxed{\cos n} \begin{array}{l} \nearrow +\infty \\ \longrightarrow -\infty \\ \not\searrow \quad \leftarrow \text{yes!} \end{array}$$

$$x_{n,2} = n \sin n \not\longrightarrow$$

$\Rightarrow$ $(\vec{x}_n)$ has not finite lim.

However, $\vec{x}_n \longrightarrow \infty_2$. To check this

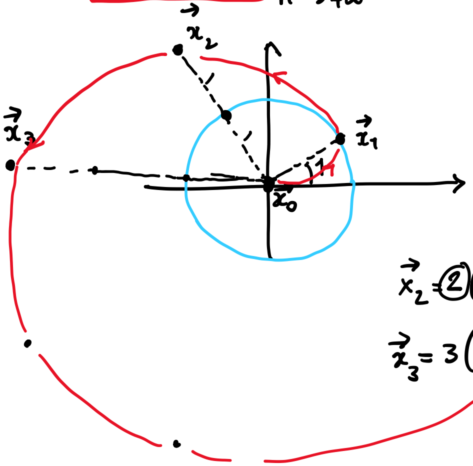
$$\boxed{\|\vec{x}_n\| \longrightarrow +\infty.}$$

$$\|\vec{x}_n\| = \sqrt{\underbrace{(n\cos n)^2 + (n\sin n)^2}}$$

$$= \sqrt{n^2(\underbrace{\cos^2 n + \sin^2 n}_{=1})}$$

$$= \sqrt{n^2} \underset{n\geq 0}{=} n$$

$$\Rightarrow \boxed{\|\vec{x}_n\| = n} \xrightarrow[n\to+\infty]{} +\infty.$$



$$\vec{x}_n = (\textcircled{n}\cos n, \textcircled{n}\sin n)$$
$$= n(\cos n, \sin n)$$

$$\vec{x}_2 = \textcircled{2}(\cos 2, \sin 2)$$
$$\vec{x}_3 = 3(\cos 3, \sin 3)$$

Do Ex 1.7.2, 1.7.3

Conclusion: we have now a def of

$$\lim_{n\to\infty} \vec{x}_n = \begin{cases} \vec{\ell} \in \mathbb{R}^d \\ \infty_d \end{cases}$$

Next Step: define limits for functs.

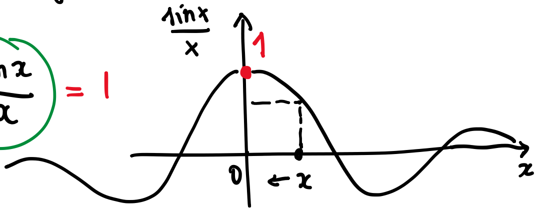
In $\mathbb{R}$: if $f = f(x) : \textcircled{D} \subset \mathbb{R} \longrightarrow \mathbb{R}$,

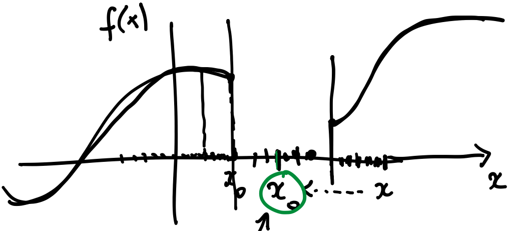
to give def of

$$\lim_{\boxed{x} \to \boxed{x_0}} \boxed{f(x)} = \ell \quad (*)$$

we need first to introduce the property x_0 must verify to have $(*)$ well defd.

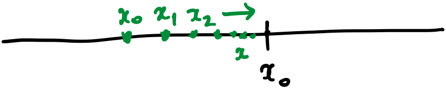
[$\lim_{x\to 0} \boxed{\frac{\sin x}{x}} = 1$





here $\lim_{x\to x_0} f(x)$ does not have any meaning

Def: (Acc. pt) We say $\boxed{x_0} \in \mathbb{R}\cup\{\pm\infty\}$ is an accumulation pt for $\boxed{D}$ if



$$\exists\ (x_n) \subset D \ : \ x_n \to x_0, \ \underline{x_n \neq x_0}$$

Def: Let $f = f(x) : D \subset \mathbb{R} \to \mathbb{R}$, $x_0 \in \underbrace{\mathrm{Acc}\, D}_{\text{set of acc. pts of } D}$

We say that

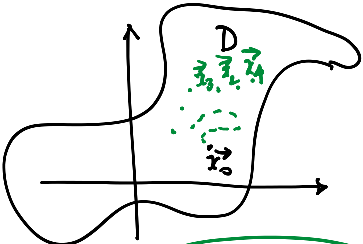
$$\lim_{\underbrace{x \to x_0}} f(x) = \ell \in \mathbb{R} \cup \{\pm\infty\}$$

if

$$\forall \underbrace{(x_n)} \subset D, \ \underline{x_n \neq x_0}, \ \underbrace{x_n \to x_0} \Rightarrow \boxed{f(x_n) \to \ell}$$

Def: Let $D \subset \mathbb{R}^d$. We say that $\vec{x}_0 \in \mathbb{R}^d \cup \{\infty_d\}$ is an accumulation point for D if

$$\exists\ (\vec{x}_n) \subset D \ : \ \vec{x}_n \neq \vec{x}_0, \ \vec{x}_n \longrightarrow \vec{x}_0$$



Def: Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0 \in \mathrm{Acc}\, D$. We say that

$$\exists \lim_{\vec{x} \longrightarrow \vec{x}_0} \underbrace{\vec{F}(\vec{x})} = \vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$$

if

$$\forall \underbrace{(\vec{x}_n)} \subset D, \ \vec{x}_n \neq \vec{x}_0, \ \underbrace{\vec{x}_n \to \vec{x}_0} \Rightarrow \boxed{\vec{F}(\vec{x}_n) \to \vec{\ell}}$$