Monday, 13 February 2023 13:00

Ex 1.7.2 Discuss $\exists$ and value of $\lim \vec{x}_n$ when

ii) $\vec{x}_n := (n, n^2)$

Claim: $\vec{x}_n \longrightarrow \infty_2 \iff \|\vec{x}_n\| \longrightarrow +\infty$

$$\|\vec{x}_n\| = \sqrt{\underbrace{n^2+n^4}_{\substack{\downarrow \\ +\infty}}} \longrightarrow +\infty. \quad \blacksquare$$

iii) $\vec{x}_n = \left(\text{Th } n, \frac{\log n}{n}, \frac{\sin n}{n}\right) \subset \mathbb{R}^3$

$$\text{Th } x = \frac{\text{Sh } x}{\text{Ch } x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{Th } n = \frac{e^n - \overset{\substack{0 \\ \uparrow}}{e^{-n}}}{e^n + \underset{\substack{\downarrow \\ 0}}{e^{-n}}} \xrightarrow[n\to+\infty]{} \boxed{1}$$

$$\frac{\log n}{n} \longrightarrow \boxed{0} \qquad n \underset{+\infty}{\gg} \log n$$

$$\frac{\sin n}{n} \xrightarrow[n\to+\infty]{} \boxed{0} \qquad \left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

$$\Rightarrow \vec{x}_n \longrightarrow (1,0,0). \quad \blacksquare$$

Ex 1.7.3

True or false?

- $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \vec{0}$

$$\Rightarrow \boxed{x_{n,j} \longrightarrow 0 \quad \bigcirc\!\!\!\!\forall\, j = 1 \dots d.}$$

TRUE because, in gen,

$$\vec{x}_n = (x_{n,1}, \dots\dots x_{n,d}) \longrightarrow \vec{\ell} \in \mathbb{R}^d$$

$$\Updownarrow$$

$$\left.\begin{array}{l} \cdot\ x_{n,1} \longrightarrow \ell_1 \\ \cdot\ x_{n,2} \longrightarrow \ell_2 \\ \quad\vdots \\ \ \ x_{n,d} \longrightarrow \ell_d \end{array}\right\} \in \mathbb{R}.$$

$$\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$$

$$\underset{?}{\Rightarrow} |x_{n,j}| \longrightarrow +\infty \quad \forall j = 1 \dots d$$

FALSE, for $\vec{x}_n = \big(\overset{\nwarrow\ \not\nearrow}{\sin n}, \overset{\nearrow +\infty}{n}\big) \longrightarrow \infty_2$

$$\vec{x}_n = (\underset{\substack{\downarrow \\ 0}}{0}, \underset{\substack{\downarrow \\ +\infty}}{n}) \longrightarrow \infty_2$$

$$\vec{x}_n = (\underset{\not\downarrow}{n\cos n}, \underset{\not\downarrow}{n\sin n}) \longrightarrow \infty_2$$

- if $\boxed{x_{n,j} \longrightarrow +\infty}$ ✓ for at least one j

(for example $x_{n,1} \longrightarrow +\infty$)

$$\overset{?}{\Longrightarrow} \vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$$

~~FALSE~~ example $\vec{x}_n = \boxed{((-1)^n, \underset{\downarrow}{n})}$

$$n \longrightarrow +\infty$$

$$\|\vec{x}_n\| = \sqrt{((-1)^n)^2 + n^2}$$

$$= \sqrt{n^2+1} \longrightarrow \boxed{+\infty}$$

$$\Rightarrow \boxed{\vec{x}_n \longrightarrow \infty_d}$$

TRUE : $\|\vec{x}_n\| = \sqrt{x_{n,1}^2 + \underbrace{x_{n,2}^2 + \dots + x_{n,d}^2}_{\geq 0}}$

$x_{n,1} \longrightarrow +\infty$ $\qquad \geq \sqrt{x_{n,1}^2}$

$$\Rightarrow \|\vec{x}_n\| \geq \sqrt{x_{n,1}^2} = |x_{n,1}| \longrightarrow +\infty$$

$$|x_{n,1}| \downarrow +\infty$$

$$\Rightarrow \|\vec{x}_n\| \longrightarrow +\infty . \quad \blacksquare$$

Do 1.7.3 the last statement.

Goal: learn how to compute a lim for a funct of vector variable.

Simplest case: $f = f(\underbrace{\overset{\mathbb{R}}{x},\overset{\mathbb{R}}{y}}_{\vec{x}}) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

Ex. 1.2.5 $\lim_{(x,y)\to \underset{\in \operatorname{Acc} D}{\overset{\vec{0}}{\boxed{(0,0)}}}} \underset{\in \mathbb{R}}{\boxed{\frac{xy}{x^2+y^2}}}$. ∃? value?

Sol: Here $f(x,y) = \frac{xy}{\boxed{x^2+y^2}} : D = \{(x,y)\in\mathbb{R}^2 ;\ x^2+y^2 \neq 0\}$

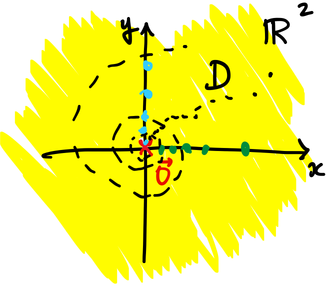
$\subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$D = \{\underbrace{(x,y)}\in\mathbb{R}^2 :\ \underbrace{x^2+y^2 \neq 0}\} = \{(x,y)\in\mathbb{R}^2 :\ (x,y)\neq\vec{0}\}$

$\underset{+}{x^2}+\underset{+}{y^2} = 0 \iff x = y = 0$

$\iff \underline{(x,y) = (0,0) = \vec{0}}$

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$



$\vec{0} \in \text{Acc}\, D$?

$\Updownarrow$

$\exists (\vec{x}_n) \subset D, \ \underline{\vec{x}_n \neq \boxed{\vec{0}}}$:

$\vec{x}_n \longrightarrow \vec{0}$

TRUE

$$\left[\ \vec{x}_n = \left(\frac{1}{n}, \frac{1}{n}\right) \longrightarrow (0,0)\right.$$

$\in D$

$$\vec{x}_n = \left(\frac{1}{n}, 0\right) \longrightarrow (0,0)$$

$\in D$

$$\left.\vec{x}_n = \left(0, 1/n\right) \longrightarrow (0,0)\ \right]$$

Ok now we know that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$xy \to 0$

makes sense

$x^2+y^2 \to 0$

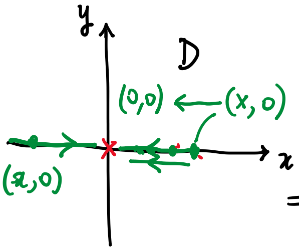
$\left(\frac{0}{0}\right)$

IND.
FORM

$(x,y) \longrightarrow (0,0) \iff \begin{matrix} x \to 0 \\ y \to 0 \end{matrix}$

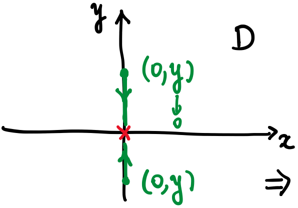
$N(x,y) = xy \longrightarrow 0$

$D(x,y) = x^2+y^2 \longrightarrow 0$



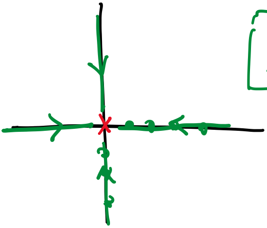
$$f(x,0) \underset{x\neq 0}{:=} \frac{x\cdot 0}{x^2+0^2} \underset{0\neq x^2}{=} \frac{0}{x^2} = 0$$

$$\Rightarrow f(x,0) \equiv \boxed{0} \longrightarrow \boxed{0}$$

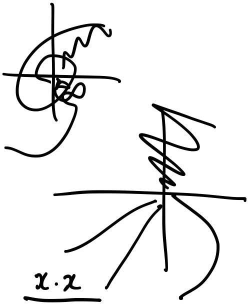


$$f(0,y) \underset{y\neq 0}{=} \frac{0\cdot y}{0^2+y^2} = \frac{0}{y^2} = 0$$

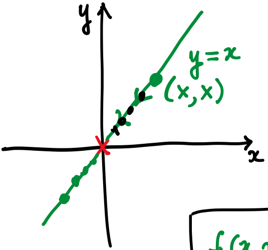
$$\Rightarrow f(0,y) \equiv 0 \longrightarrow 0$$



$$\boxed{f \longrightarrow \boxed{0}}$$



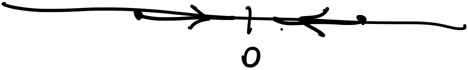
Is 0 the limit of f ?



$$f(x,x) \underset{x\neq 0}{=} \frac{x\cdot x}{x^2+x^2} = \frac{1\cancel{x^2}}{2\cancel{x^2}} \equiv 1/2$$

$$\boxed{f(x,x) \equiv 1/2} \longrightarrow \boxed{1/2}$$

$$\Rightarrow \quad \nexists \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}. \qquad \blacksquare$$



Ex 1.2.6. Discuss existence and value (if any) of

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

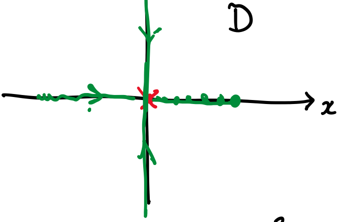
Sol: Here $f(x,y) = \dfrac{xy^2}{x^2+y^4} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

where $D = \{(x,y)\in\mathbb{R}^2 : x^2+y^4 \neq 0\}$

and since $\underbrace{x^2}_{+} + \underbrace{y^4}_{+} = 0 \Leftrightarrow x=y=0$

$\Leftrightarrow (x,y) = \vec{0}$

$\Rightarrow \quad D = \mathbb{R}^2 \setminus \{\vec{0}\}$ $\qquad f(x,0) = \dfrac{x\cdot 0^2}{x^2+0^4} =$

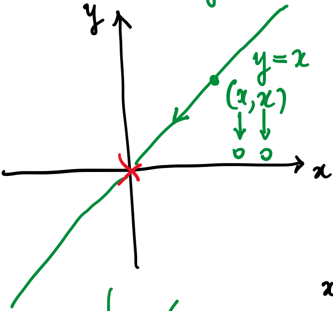


$$= \frac{0}{x^2} \equiv 0$$

$$f(x,0) \equiv 0 \xrightarrow[x \to 0]{} \boxed{0}$$

$$f(0,y) = \frac{0 \cdot y^2}{0^2 + y^4} = \frac{0}{y^4} \equiv 0$$

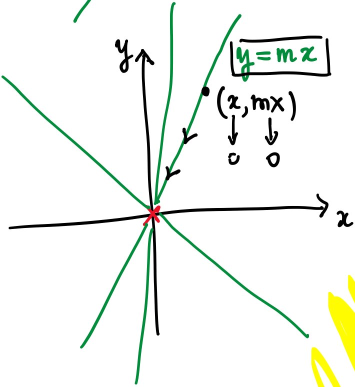
$$\to f(0,y) \equiv 0 \xrightarrow[y \to 0]{} \boxed{0}$$



$$f(x,x) = \frac{x \cdot x^2}{x^2 + x^4}$$

$$= \frac{x^3}{\boxed{x^2} + x^4} = \frac{x}{1 + x^2}$$

$$\xrightarrow[x \to 0]{} \boxed{0}$$



$$\boxed{y = mx} \qquad m \in \mathbb{R}$$

$$f(x,mx) =$$

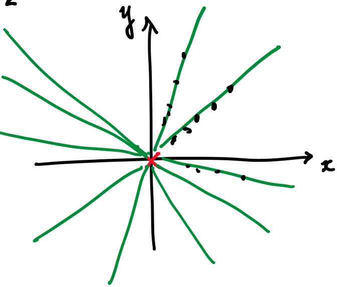
$$= \frac{x\cdot(mx)^2}{x^2+(mx)^4}$$

$$= \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x^{\cancel{3}}}{\cancel{x^2}(1+m^4x^2)}$$

$$= \frac{m^2x}{1+m^4x^2} \xrightarrow[x\to 0]{} \boxed{0}$$

Conclusion: no matter how we go to $\vec{0}$ along straigth lines $\boxed{y = mx}$

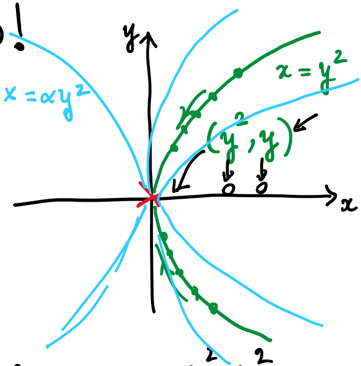
$f \longrightarrow \boxed{0}$



Question: is that sufficient to say that

$$\exists \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = 0 \ ?$$

NO!



$$f(x,y) = \boxed{\frac{xy^2}{x^2+y^4}}$$

$$x = y^2$$

$$f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{1\cancel{y^4}}{2\cancel{y^4}}$$

$$\equiv 1/2 \longrightarrow \boxed{1/2} \neq 0.$$

$$(x = \alpha y^2)$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \qquad \blacksquare$$

GEN RMK: What we learnt here is that if there "two ways" to go to $\vec{x}_0$ such that sections

on the first way $f \longrightarrow \ell_1$

on the second way $f \to \ell_2^{\#}$

$$\Rightarrow \quad \nexists \lim_{\vec{x}\to\vec{x}_0} f(\vec{x}).$$

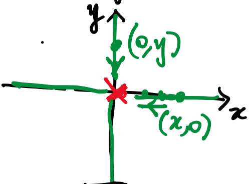
Do Ex 1.7.6

#1. Looking at suitable sections, prove that the following lim does not ∃

$$\lim_{(x,y)\to(0,0)} \boxed{\frac{x^2-y^2}{x^2+y^2}}.$$

Sol: Here $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ dcfd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$.



We have

$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \longrightarrow \boxed{1}$$

$$\neq$$

$$f(0,y) = \frac{-y^2}{y^2} \equiv -1 \longrightarrow \boxed{-1}$$

$$\neq$$

$$\left(\quad f(x,x) = \frac{0}{2x^2} \equiv 0 \longrightarrow \boxed{0} \right)$$

Do #2, #3, #5

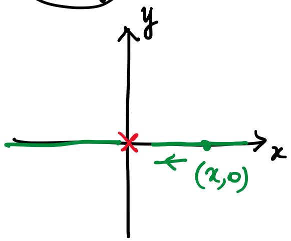
Ex 1.2.8 Discuss existence and value

of

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}.$$

Sol: Here $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ defd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$



Since $(x,y) \to (0,0)$

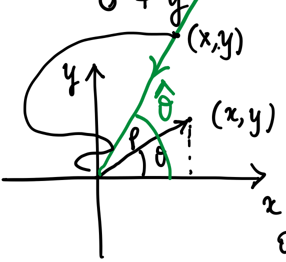
$\Rightarrow$ $x \to 0$
$y \to 0$

$\Rightarrow$ $N = xy^2 \to 0$
$D = x^2+y^2 \to 0$

$\left(\frac{0}{0}\right)$ I.F.

$$f(x,0) = \frac{x\cdot 0^2}{x^2+0^2} = \frac{0}{x^2} \equiv 0 \xrightarrow{x\to 0} \boxed{0}$$

$$f(0,y) = \frac{0\cdot y^2}{0^2+y^2} \equiv 0 \xrightarrow[y\to 0]{} \boxed{0}$$



$$\boxed{\boxed{\rho} \downarrow 0} = \sqrt{x^2+y^2} = \|(x,y)\| \downarrow 0$$

$0 \leq \theta \leq 2\pi$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\boxed{x^2+y^2} = (\rho\cos\theta)^2 + (\rho\sin\theta)^2 = \rho^2\underbrace{\left[\cos^2\theta+\sin^2\theta\right]}_{1} \boxed{= \rho^2}$$

$$f(x,y) = f(\rho\cos\theta, \rho\sin\theta)$$

$$= \frac{\rho\cos\theta\cdot(\rho\sin\theta)^{2}}{\rho^2}$$

$$= \frac{\rho^{\cancel{3}}\cos\theta\sin^2\theta}{\cancel{\rho^2}}$$

$$\underbrace{f(x,y) = \underset{0}{\overset{\downarrow}{\rho}}\cdot\overset{\downarrow}{\boxed{\underset{-1\leq\ \ \leq1}{\cos\theta}\cdot\underset{0\leq\ \ \leq1}{\sin^2\theta}}}}_{}$$

borné

$$\|(x,y)-(0,0)\| \to 0 \iff (x,y)\to\vec{0}$$

$$\|(x,y)\| \to 0$$

$$-\rho \leq f(x,y) \leq \rho\cdot1\cdot1 = \rho$$

$$\underset{0}{\overset{}{-\|(x,y)\|}} = \underset{0}{\boxed{-\rho \leq f(x,y) \leq \rho}} = \underset{\boxed{0}}{\|(x,y)\|}$$

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} f = 0. \qquad \blacksquare$$

Do Ex 1.7.7 #1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.