Tuesday, 14 February 2023 13:03

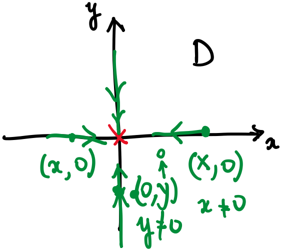
Ex 1.7.6

By Using suitable sections, prove that the following lim does not ∃

#2. $$\lim_{(x,y)\to\underline{(0,0)}} \frac{x^2+y^3}{x^2+y^2}.$$

Sol: Here $f(x,y) := \dfrac{x^2+y^3}{x^2+y^2}$, $(x,y)\in D$

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$



$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x\to 0]{} \boxed{1}$$

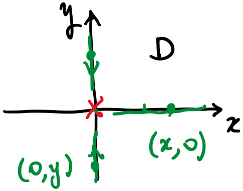
$$f(0,y) = \frac{y^3}{y^2} = y \xrightarrow[y\to 0]{} \boxed{0}$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y) \quad \blacksquare$$

#3. $$\lim_{(x,y)\to(0,0)} \frac{y^2-xy}{x^2+y^2}.$$

Sol: Here $f(x,y) := \frac{y^2 - xy}{x^2 + y^2}$ defd on

$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus \{\vec{0}\}$



$$f(x,0) = \frac{0}{x^2} \equiv 0 \xrightarrow[x \to 0]{} \boxed{0}$$

$$\#$$

$$f(0,y) = \frac{\cancel{y^2}}{\cancel{y^2}} \equiv 1 \xrightarrow[y \to 0]{} \boxed{1}$$

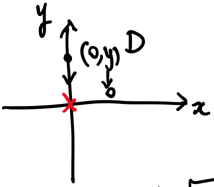
$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y) \quad \blacksquare$$

#5 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy + \sqrt{y^2+1} - 1}{x^2+y^2}$.

Sol: Here $f(x,y) := \frac{xy + \sqrt{y^2+1} - 1}{x^2+y^2}$

defd on $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\}$

$= \mathbb{R}^2 \setminus \{\vec{0}\}$.



$$f(x,0) = \frac{0+\sqrt{1}-1}{x^2} = \frac{0}{x^2} = 0 \xrightarrow{x\to 0} \boxed{0}$$

$$f(0,y) = \frac{\sqrt{\underbrace{y^2+1}_{\to 1}}-1}{\underset{\to 0}{y^2}} \to \frac{0}{0} \quad y\to 0$$

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0}\frac{\sqrt{y^2+1}-1}{y^2} = \frac{(1+y^2)^{1/2}-1}{y^2} = \boxed{1/2}$$

$$\boxed{0} \neq \boxed{1/2}$$

$$\left(\lim_{t\to 0}\frac{(1+t)^{\alpha}-1}{t} = \alpha \right.$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y). \quad \blacksquare$$

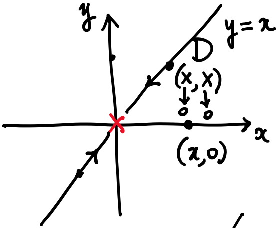
Ex 1.7.7.

$$\#1 \quad \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

(Compute the limit)

Sol: Here $f(x,y) := \frac{xy}{\underbrace{\sqrt{x^2+y^2}}_{\geqslant 0}}$. defd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$



- $f(x,0) = \frac{0}{\sqrt{x^2}} = 0 \xrightarrow[x\to 0]{} \boxed{0}$

- $f(0,y) \equiv 0 \xrightarrow[y\to 0]{} \boxed{0}$

$$f(x,x) = \frac{x^2}{\sqrt{x^2+x^2}} = \frac{x^2}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}}\,\frac{x^2}{|x|}$$

$$= \frac{1}{\sqrt{2}}\boxed{\frac{x}{|x|}} \cdot x$$

$$\llcorner = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$$

$$= \frac{1}{\sqrt{2}}\underbrace{\boxed{(\operatorname{sgn} x)}}_{\text{bounded}} \cdot \underset{\substack{\downarrow \\ 0}}{x} \xrightarrow[x\to 0]{} \boxed{0}$$

$$\nexists \lim_{x\to 0} \operatorname{sgn} x$$

$$\lim_{x\to 0+} \overset{+1}{\overset{\equiv}{\operatorname{sgn}}}\, x = +1$$

$$\neq$$

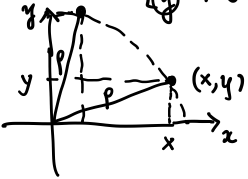
$$\lim_{x\to 0-} \overset{-1}{\overset{\equiv}{\operatorname{sgn}}}\, x = -1$$

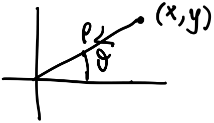
We now try to prove $\exists$ of lim:

$$\lim_{(x,y)\to \underbrace{(0,0)}} \frac{\overset{\to 0}{\boxed{xy}}}{\sqrt{\underbrace{x^2+y^2}_{\downarrow\, 0}}} \to 0 \qquad \frac{0}{0}$$

$$\Updownarrow$$

$$\begin{cases} \boxed{x} \to 0 \\ \boxed{y} \to 0 \end{cases}$$





$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad x^2+y^2=\rho^2$$

$$\rho = \|(x,y)\| \to 0$$

$$f(x,y) = \frac{\rho\cos\theta\cdot\rho\sin\theta}{\sqrt{\rho^2}} \qquad \sqrt{\rho^2} = |\rho| \overset{\rho\geq 0}{=} \rho$$

$$= \frac{\rho^{\cancel{2}}}{\ } \cos\theta\cdot\sin\theta$$

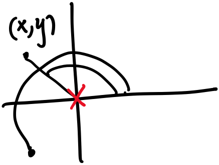
$$f(x,y) = \underset{\substack{\downarrow \\ 0}}{\overset{\cancel{\rho}}{\boxed{\rho}}} \cdot \underset{-1\le \qquad\qquad \le 1}{\overset{\substack{0\le\theta\le 2\pi \\ 1}}{\boxed{\cos\theta \sin\theta}}} = \frac{1}{2}\sin(2\theta)$$

$$\boxed{-\rho} \le f(x,y) \le 1\cdot\rho = \underset{\|}{\boxed{\rho}}$$

$$\underset{\substack{\downarrow \\ 0}}{-\|(x,y)\|} \le \underset{\substack{\downarrow \\ 0}}{f(x,y)} \le \underset{\substack{\downarrow \\ 0}}{\|(x,y)\|} \qquad (x,y)\to\vec{0}$$

by the police thm.

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0. \quad \blacksquare$$

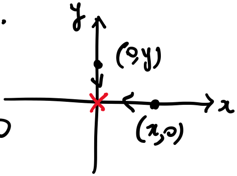


Ex 1.7.7

#2. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2}.$$

Sol: Here $f(x,y) := \dfrac{x^2y^3}{(x^2+y^2)^2}$ is

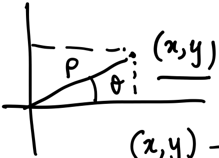
def'd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.



Sections:

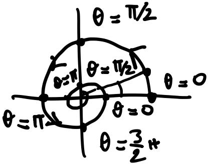
$$f(x,0) = \frac{0}{x^4} \equiv 0 \xrightarrow[x\to 0]{} 0$$

$$f(0,y) = \frac{0}{y^4} \equiv 0 \xrightarrow[y\to 0]{} 0$$



$$\rightsquigarrow \boxed{(\rho,\theta)}$$

$$(x,y) \to \vec{0} \iff \begin{cases} x \to 0 \\ y \to 0 \end{cases}$$



$$\iff \overset{?}{\boxed{\rho \to 0}} \quad \rho, \theta$$

$$f(x,y) \underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \frac{\rho^2(\cancel{\rho}\cos\theta)^2\rho^3(\cancel{\rho}\sin\theta)^{\cancel{3}}}{(\rho^2)^2} \leftarrow$$

$$= \frac{\rho^{5} (\cos\theta)^{2} \cdot (\sin\theta)^{3}}{\rho^{4}}$$

$$f(x,y) = \underset{\substack{\downarrow \\ 0}}{\boxed{\rho}} \cdot \underbrace{\underbrace{(\cos\theta)^{2}}_{\substack{\le 1 \\ \ge 0}} \cdot \overbrace{\underbrace{(\sin\theta)^{3}}_{\substack{\le 1 \\ \ge -1}}}^{\ge -1}}_{\substack{\le 1 \\ \ge -1}} \leftarrow$$

$$\underset{\substack{\downarrow \\ 0}}{-\|(x,y)\|} = \boxed{\underset{-\varphi(\rho)}{-\rho} \le \underset{\substack{\downarrow \\ 0}}{f(x,y)} \le \underset{\varphi(\rho)}{\rho}} = \underset{\substack{\downarrow \\ 0}}{\|(x,y)\|} \qquad \Rightarrow \quad \Leftarrow \qquad \boxed{(x,y) \to \vec{0}}$$

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} f(x,y) = 0. \quad ■$$

Do Ex 1.7.7 # 3,4,5.

Rmk: What if I use pol. coords in a case where lim does not ∃?

1.7.6

#2 $\lim\limits_{(x,y)\to\vec{0}} \dfrac{x^2+y^3}{x^2+y^2}$.

Here $f(x,y) =$

$f(x,y)$ $\quad x = \rho\cos\theta$
$\quad \underline{\underline{y}} = \rho\sin\theta$

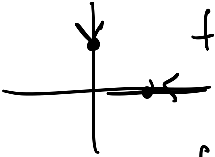
$$= \frac{\cancel{\rho^2}(\cos\theta)^2 + \cancel{\rho^3}(\sin\theta)^3}{\cancel{\rho^2}}$$

$$= \underbrace{(\cos\theta)^2} + \boxed{\underbrace{\rho}_{\downarrow 0}\underbrace{(\sin\theta)^3}_{-1\leq \quad \leq 1}} \to 0$$

$\boxed{(x,y)\to\vec{0}} \Longleftrightarrow \boxed{\rho\to 0}$

$0 \leq \theta \leq 2\pi$

$\theta = \pi/2$



$f(0,y) = \dfrac{y^3}{y^2} \equiv y \to \boxed{0}$

$\theta = 0$

$f(x,0) = \dfrac{x^2}{x^2} \equiv 1 \to \boxed{1}$

■

Ex $\quad f(x,y) = \dfrac{xy^2}{x^2+y^4}$

(we already checked that $\nexists \lim_{(x,y)\to\vec{0}} f$)

$$f(x,y) \overset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \frac{\rho\cos\theta\,(\rho\sin\theta)^2}{(\rho\cos\theta)^2+(\rho\sin\theta)^4}$$

$$= \frac{\rho^{\cancel{3}}\cos\theta\,(\sin\theta)^2}{\cancel{\rho^2}(\cos\theta)^2+\rho^{\overset{2}{\cancel{4}}}(\sin\theta)^4}$$

$$= \frac{\overset{\overset{0}{\uparrow}}{\boxed{\rho}}\ \overset{0}{\boxed{\overset{//}{\cos\theta}\ (\sin\theta)^2}}}{\underbrace{\boxed{(\cos\theta)^2}}_{\overset{\downarrow}{0}} + \underbrace{\rho^2(\sin\theta)^{\overset{4}{}}}_{\overset{\downarrow}{0}} \searrow 0} \longrightarrow 0$$

$$\theta = \boxed{\pi/2}$$

$$\downarrow$$

$$f\underset{\sim}{(0,y)} = \frac{0}{y^4} \equiv 0 \longrightarrow 0$$

General existence rule:

In prev examples we did

Step 1: change (x,y) with $(\rho\cos\theta, \rho\sin\theta)$

$$f(x,y) = \underbrace{f(\rho\cos\theta, \rho\sin\theta)}$$

Step 2: Show that

$$\underset{\substack{\downarrow \\ 0}}{\boxed{-\varphi(\rho)}} \le \underbrace{f(\rho\cos\theta, \rho\sin\theta)} \le \underset{\substack{\downarrow \\ 0}}{\boxed{\varphi(\rho)}}$$

st. $\varphi(\rho) \to 0$ when $\rho \overset{o}{\to} 0$

Step 3: (conclusion)

$$\boxed{\exists \lim_{(x,y)\to\vec{0}} f(x,y) = 0}$$

Ex: $$\lim_{(x,y)\to\vec{0}} \frac{x^2y + y^2x}{\sqrt{\underbrace{x^2+y^2}_{\not> 0}}}$$

Sol: Here $f(x,y) :=$ is defd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

In polar coords we have

$$f(x,y) \overset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \frac{(\rho\cos\theta)^2(\rho\sin\theta) + (\rho\sin\theta)^2(\rho\cos\theta)}{\sqrt{\rho^2}} \quad \sqrt{\rho^2} = |\rho| = \rho, \ \rho \geq 0$$

$$= \underline{\rho^{\overset{2}{\cancel{3}}}\left[(\cos\theta)^2\sin\theta + (\sin\theta)^2\cos\theta\right]}$$

$\cancel{\rho}$

$(x,y)\to\vec{0}$

$\Updownarrow$

$\rho=\|(x,y)\|\to 0$

$$f = \underset{\downarrow \atop 0}{\boxed{\rho^2}}\cdot\overbrace{\underbrace{\Big[\underbrace{(\cos\theta)^2}_{\le 1}\underbrace{\sin\theta}_{\le 1} + \underbrace{(\sin\theta)^2\cos\theta}_{\le 1}\Big]}_{}}^{\le 2}$$

$$\underbrace{\le 1}_{} \qquad \le 1$$

$$\underbrace{\ge -1 \qquad \ge -1}_{\ge -2}$$

$$\underset{-\varphi(\rho)}{-2\rho^2} \le \underline{f(x,y)} \le \underbrace{\boxed{2\rho^2}}_{\varphi(\rho)} \to 0 \qquad \rho\to 0$$

$$\underset{0}{\downarrow} \quad \Rightarrow \quad \underset{0}{\downarrow} \quad \Leftarrow \quad \underset{0}{\downarrow}$$

$$\Rightarrow\ \exists \lim_{(x,y)\to\vec{0}} f(x,y) = 0. \qquad \blacksquare$$

Rmk: (extension of the prev rule)

What if I have to prove that

$$\lim_{(x,y)\to\vec{0}} \underbrace{f(x,y)}_{} = \underbrace{\ell}_{} \in \mathbb{R}$$

$\Updownarrow$

$$\lim_{(x,y)\to\vec{0}} \underbrace{\left(f(x,y)-\ell\right)}_{} = 0$$

If

$$\boxed{\underset{\underset{0}{\downarrow}}{-\varphi(\rho)} \le \underbrace{f(x,y)-\ell}_{} \le \underset{\underset{0}{\downarrow}}{\varphi(\rho)}}$$

$\varphi(\rho) \to 0$ when $\rho \to 0$

$$\Rightarrow \lim_{(x,y)\to\vec{0}} f(x,y) = \ell.$$

Ex 1.7.8

#1 $$\lim_{(x,y)\to\vec{0}} \boxed{\frac{\underbrace{e}_{}^{4y^3} - \boxed{\cos}(x^2+y^2)}{x^2+y^2}}$$

Sol: Here $f(x,y)$ clearly dfd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$. Notice that

$$(x,y) \to \vec{0} \Rightarrow \begin{array}{l} x \to 0 \\ y \to 0 \end{array}$$

$$e^{4y^3 \to 0} \longrightarrow 1$$

$$\cos(\underbrace{x^2+y^2}_{\underset{0}{\downarrow}}) \longrightarrow 1$$

$$\Rightarrow N = e^{4y^3} - \cos(x^2+y^2) \longrightarrow 0$$

$$D = x^2+y^2 \to 0$$

so lim is a $0/0$ I.F.

Let's write f in pol coords

$$f(x,y) = \frac{e^{\boxed{4\rho^3\sin^3\theta} \to 0} - \cos\rho^2 \to 0}{\rho^2}$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$\frac{e^{\boxed{ⓐ\rho^3}} - \cos ⓟ^2}{\rho^2} \qquad \boxed{\rho \to 0}$$

$$e^t = \boxed{1 + t + o(t)} = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$\cos t = \boxed{1 - \frac{t^2}{2} + o(t^2)} = 1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^4)$$

$$= \frac{\cancel{1} + 4\rho^3 + \boxed{o(\rho^3)} - \left(\cancel{1} - \cancel{\frac{\rho^4}{2}} + \cancel{\boxed{o(\rho^4)}}\right.}{\rho^2}$$

$$= \frac{4\rho^3 + o(\rho^3)}{\rho^2} = 4\rho + o(\rho). \xrightarrow[\rho \to 0]{} 0.$$

Returning to f t

$$\underline{f(x,y)} = \frac{e^{\boxed{④\rho^3(\sin\theta)^3}} - \cos\boxed{\rho^2}}{\rho^2}$$

$$= \frac{\cancel{1}+4\rho^3(\sin\vartheta)^3+\boxed{o(\rho^3)}-\left(\cancel{1}-\cancel{\frac{\rho^4}{2}}+\cancel{o(\rho^4)}\right)}{\rho^2}$$

$e^t = 1+t+o(t)$, $\cos t = 1-\frac{t^2}{2}+o(t^2)$

$$= \frac{4\rho^{\cancel{3}}(\sin\vartheta)^3+o(\rho^3)}{\cancel{\rho^2}}$$

$$\boxed{f(x,y) = \underset{\downarrow \atop 0}{\boxed{4\rho(\sin\vartheta)^3}} + \underset{\downarrow \atop 0}{o(\rho)}} \qquad \rho\to 0$$

$$\begin{cases} x=\rho\cos\vartheta \\ y=\rho\sin\vartheta \end{cases}$$

Guess: $\ell = 0$

$$\underbrace{-4\rho+o(\rho)}_{\downarrow \atop 0} \le \underset{\downarrow \atop 0}{f(x,y)} - \underset{\parallel \atop 0}{\ell} \le \underbrace{4\rho+o(\rho)}_{\downarrow \atop 0}$$

$$0 \Rightarrow 0 \Leftarrow 0$$

$$\Rightarrow \exists \lim_{(x,y)\to\vec{0}} f = 0 \quad ■$$

Do Ex 1.7.7 #4, #5, #6.

Rmk: What if I have to compute

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(u,v)\to(0,0)} f(u+x_0, v+y_0)$$

$$\begin{cases} u=x-x_0 \\ v=y-y_0 \end{cases}$$

$$(x,y)\to(x_0,y_0) \iff (x-x_0,\, y-y_0)\to(0,0)$$

$$\Updownarrow$$

$$\begin{matrix} x\to x_0 \\ y\to y_0 \end{matrix} \iff \begin{matrix} x-x_0\to 0 \\ y-y_0\to 0 \end{matrix} \qquad \begin{matrix} x=u+x_0 \\ y=v+y_0 \end{matrix}$$