Thursday, 16 February 2023 13:05

Computing Limits

1.7.7

#3 $\lim\limits_{\underbrace{(x,y)\to(0,0)}}$

$$\frac{\overset{\nearrow 0}{x^3} - \overset{\nearrow 0}{y^3} \to 0}{x^2+y^2 \to 0}$$

Here $f(x,y) := \dfrac{x^3-y^3}{x^2+y^2}$ defd on

$D = \mathbb{R}^2 \setminus \{\vec{0}\}$. In pol. coords

$$f(x,y) \underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \frac{\rho^3\cos^3\theta - \rho^3\sin^3\theta}{\rho^2}$$

$$= \frac{\rho^{\cancel{3}}}{\cancel{\rho^2}}\left(\cos^3\theta - \sin^3\theta\right)$$

$$f(x,y) = \underset{\substack{\downarrow\\ 0}}{\boxed{\rho}}\ \underbrace{\boxed{\left(\overset{-1\le\ \ \le 1}{\cos^3\theta} - \overset{-1\le\ \ \le 1}{\sin^3\theta}\right)}}_{-2\le\ \ \ \le 2}$$

$\left(\rho = \|(x,y)\| \longrightarrow 0 \text{ because } (x,y)\to\vec{0}\right)$

$$\Downarrow$$

$$\underset{\substack{\downarrow\\ 0}}{-2\rho} \le \underset{\substack{\downarrow\\ \ }}{f(x,y)} \le \underset{\substack{\downarrow\\ 0}}{2\rho}$$

by police thm.^{0} $\Rightarrow \exists \lim_{(x,y)\to\vec{0}} f = 0$ ■

#4

$$\lim_{(x,y)\to\vec{0}} \frac{x\sqrt{|y|}}{\sqrt[3]{\underline{x^4+y^4}}}$$

(do this after sol of #5)

#5

$$\lim_{(x,y)\to\vec{0}} \frac{\overset{\to 0}{\boxed{xy}}}{\underbrace{|x|+|y|}_{\downarrow\, 0}} . \qquad \frac{0}{0}$$

Here $f(x,y) = \dfrac{\boxed{xy}}{\boxed{|x|+|y|}}$ is defd on

$D = \{(x,y)\in\mathbb{R}^2 : |x|+|y| \neq 0\}$. Now

$$\underbrace{|x|}_{+} + \underbrace{|y|}_{+} = 0 \iff x = y = 0$$

$$\Updownarrow$$

$$\iff |x| = |y| = 0$$

$\Rightarrow \quad D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

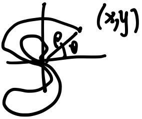
Claim: lim ∃. Let's see f in pol. coords

$$f(x,y) = \frac{\rho^2 \cos\theta \sin\theta}{|\rho\cos\theta| + |\rho\sin\theta|}$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$= \frac{\rho^{\cancel{2}} \cos\theta \sin\theta}{\cancel{\rho}\,(|\cos\theta| + |\sin\theta|)}$$



$$\to f = \underset{\downarrow\, 0}{\boxed{\rho}}\ \boxed{\frac{\cos\theta \ \sin\theta}{|\cos\theta| + |\sin\theta|}}$$

$$\boxed{\rho = \|(x,y)\| \to 0}$$

$0 \le \theta \le 2\pi$

$\boxed{-K} \le$? $\boxed{\le K}$ indep of $\underline{\theta}$

$\Downarrow$?

$$\boxed{-K\rho \le f \le K\rho}$$

$$\boxed{\left| \ — \ \right| \le K}$$

$-K\rho \to 0$, $f \to 0$, $K\rho \to 0$

$$\Rightarrow \exists \lim_{(x,y)\to\vec{0}} f = 0 ,$$

$$0 \le \left| \frac{\cos\theta \sin\theta}{|\cos\theta| + |\sin\theta|} \right| = \frac{0 \le \boxed{\overset{\le 1}{|\cos\theta|}\ \overset{\le 1}{|\sin\theta|}} \le 1}{\boxed{0 \le}\ \underbrace{|\cos\theta| + |\sin\theta|}\ \boxed{\le 2}} \ \boxed{\le} +\infty$$

$\nexists K$?

$a \le 1$

$$\frac{0 \le \boxed{a} \le 1}{\boxed{0 \le b \le 2}} \le \quad \frac{1}{b} \le \frac{1}{0} \ ?$$

$0 \leqslant b \leqslant 2 \qquad \frac{1}{2} \leqslant \frac{1}{b} < +\infty$

$\uparrow$

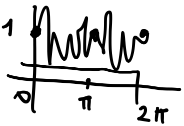
$$\left|\frac{\cos\theta\ \sin\theta}{|\cos\theta|+|\sin\theta|}\right| = \frac{\overbrace{|\cos\theta||\sin\theta|}^{\leqslant 1}}{|\cos\theta|+|\sin\theta|} \leqslant \frac{\textcircled{1}}{|\cos\theta|+|\sin\theta|} \overset{?}{\leqslant} k$$

$\cancel{\geqslant}\,\textcircled{0} = \frac{1}{k}$ No!

$$\Longleftrightarrow \boxed{\frac{1}{|\cos\theta|+|\sin\theta|} \leqslant k} \Longleftrightarrow \boxed{|\cos\theta|+|\sin\theta| \geqslant \frac{1}{k}}$$

$\underline{\theta \in [0,2\pi]}$

$$\textcircled{g(\theta)} = \boxed{\underbrace{|\cos\theta|}_{\overset{+}{?}}+\underbrace{|\sin\theta|}_{+}} \geqslant 0$$



$$\boxed{g(\theta) = 0}$$

$$\Longleftrightarrow \begin{cases} |\cos\theta| = 0 \\ |\sin\theta| = 0 \end{cases} \Longleftrightarrow \begin{cases} \cos\theta = 0 \\ \sin\theta = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \theta = \pi/2,\ \frac{3}{2}\pi \\ \theta = 0, \pi, 2\pi \end{cases}$$ NO θ.

$\Rightarrow \underline{g(\theta) > 0} \quad \underline{\forall\, \theta \in [0,2\pi]}$

Now since $g \in \mathcal{C}([0,2\pi])$

$\uparrow$

closed & bded

$\Rightarrow$ (W. thm) $\exists \; \boxed{\min\limits_{[0,2\pi]} g}, \; \max\limits_{[0,2\pi]} g$

$$\exists\, \theta^* \in [0,2\pi] \; : \; \boxed{g(\theta)} \geq \boxed{\overset{>0}{g(\theta^*)}} \quad \forall \theta \in [0,2\pi]$$

$$g(\theta^*) = \frac{1}{K} \qquad K = \frac{1}{g(\theta^*)}$$

$$\Rightarrow |\cos\theta| + |\sin\theta| \geq \frac{1}{K} \quad \forall \theta \in [0, 2\pi]$$

$\Rightarrow$ from this the conclusion follows. ■

# 4 $\displaystyle \lim_{(x,y)\to\vec{0}} \frac{x\sqrt{|y|}}{\sqrt[3]{\underset{\geq 0}{x^4+y^4}}} \overset{?}{=} 0 \qquad \left(\frac{0}{0}\right)$

Sol: Here $f(x,y) :=$ defd

$$D = \{x^4+y^4 \neq 0\} = \mathbb{R}^2 \setminus \{\vec{0}\}$$

In pol. coords

$$f(x,y) = \frac{\rho\cos\theta\, \rho^{1/2}\sqrt{|\sin\theta|}}{\left(\rho^4(\cancel{\rho}\cos\theta)^4 + \rho^4(\cancel{\rho}\sin\theta)^4\right)^{1/3}}$$

$$= \frac{\rho^{3/2}}{\rho^{4/3}} \; \frac{\cos\theta\sqrt{|\sin\theta|}}{\left((\cos\theta)^4 + (\sin\theta)^4\right)^{1/3}}$$

$$= \rho^{3/2 - 4/3} \frac{\cos\theta \sqrt{|\sin\theta|}}{\left((\cos\theta)^4 + (\sin\theta)^4\right)^{1/3}}$$

$$\frac{3}{2} - \frac{4}{3} = \frac{9-8}{6} = 1/6$$

$$(x,y) \to \vec{0}$$
$$\Downarrow$$
$$\rho = \|(x,y)\| \to 0$$

$$\hookrightarrow = \underset{\substack{\downarrow \\ 0}}{\rho^{1/6}} \cdot \boxed{\frac{\cos\theta \sqrt{|\sin\theta|}}{\left((\cos\theta)^4 + (\sin\theta)^4\right)^{1/3}}}$$

$\theta \in [0, 2\pi]$

$$\left| \frac{\cos\theta \sqrt{|\sin\theta|}}{\left((\cos\theta)^4 + (\sin\theta)^4\right)^{1/3}} \right| \overset{?}{\leq} K \quad \forall \theta \in [0, 2\pi]$$

Yes, do the details himitating prev exercise.

$$\begin{tikzcd} \Rightarrow \quad -K\rho^{1/6} \arrow[d] & \leq f \arrow[dd] & \leq K\rho^{1/6} \arrow[d] \\ 0 & & 0 \\ & 0 & \end{tikzcd}$$

by Police thm

$$\Rightarrow \exists \lim_{(x,y) \to \vec{0}} f(x,y) = 0. \quad \blacksquare$$

Limits at ∞_d

So far we introduced the operation of

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = \ell \in \mathbb{R}\cup\{\pm\infty\}$$

and $\vec{x}_0 \in \mathbb{R}^d$. The def says:

$\forall (\vec{x}_n) \in D,\ \vec{x}_n \neq \vec{x}_0 \ :\ \vec{x}_n \to \vec{x}_0 \Rightarrow f(\vec{x}_n) \to \ell$

Now we want to treat

$$\lim_{\vec{x}\to\boxed{\infty_d}} f(\vec{x}) = \ell \in \mathbb{R}\cup\{\pm\infty\}.$$

__Def:__ Let $f = f(\vec{x}) : D \subset \underline{\mathbb{R}^d} \to \boxed{\mathbb{R}}$, $\infty_d \in \operatorname{Acc} D$.

$(\Leftrightarrow\ \exists\ (\vec{x}_n) \subset \boxed{D} : \underline{(\cancel{\vec{x}_n \neq \infty_d})}\ \vec{x}_n \to \infty_d) \leftarrow$

We say that

$$\exists \lim_{\vec{x}\to\infty_d} f(x) = \ell \in \mathbb{R}\cup\{\pm\infty\}$$

if

$\forall (\vec{x}_n) \subset D,\ \underline{(\cancel{\vec{x}_n \neq \infty_d})} : \vec{x}_n \to \infty_d \Rightarrow f(\vec{x}_n) \to \ell.$

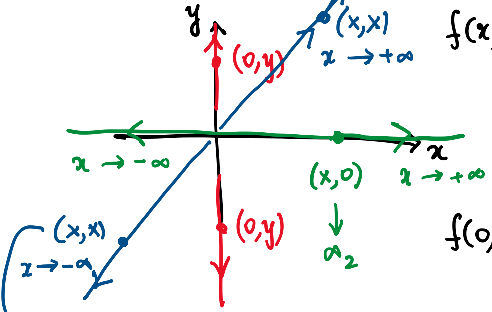
__Ex.__ Discuss existence and value of

$$\lim_{(x,y)\to\infty_2} xy$$

__Sol:__ Here $f(x,y) = xy$ is defd on $\underline{D = \mathbb{R}^2}$

(notice that, clearly, $\underline{\infty_2} \in \text{Acc}\,\textcircled{D}$ because

$\vec{x}_n := (n,n) \subset D$, $\vec{x}_n \to \infty_2$)



$$f(x,0) = x \cdot 0 = 0 \xrightarrow[x\to-\infty]{x\to+\infty} \boxed{0}$$

$$f(0,y) = 0 \cdot y \equiv 0 \xrightarrow{y\to\pm\infty} \boxed{0}$$

$$(x,x) \to \infty_2 \iff \|(x,x)\| \to +\infty$$

$$\|(x,x)\| = \sqrt{x^2+x^2} = \sqrt{2}\,\textcircled{|x|} \iff |x| \to +\infty$$

$$\iff x \to +\infty \ \vee\ x \to -\infty$$

$$f(x,x) = x \cdot x = x^2 \xrightarrow[x\to\pm\infty]{} \boxed{+\infty}$$

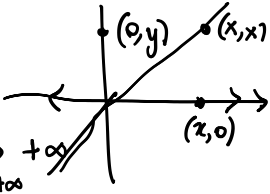
$$\Rightarrow \nexists \lim_{(x,y)\to\infty_2} f(x,y). \ \blacksquare$$

Ex: Discuss $\exists$ and value (if any)

of
$$\lim_{(x,y)\to\infty_2} \underline{(x^2+y^2-4xy)}$$

Sol: Here $f(x,y) := x^2+y^2-4xy$ is défd on

$D = \mathbb{R}^2$.



Sections:

$$f(x,0) = x^2 \xrightarrow[x\to\pm\infty]{} +\infty$$

$$f(0,y) = y^2 \xrightarrow[y\to\pm\infty]{} \boxed{+\infty}$$

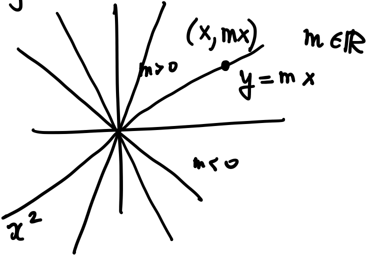
$$f(x,x) = 2x^2 - 4x^2 = -2x^2 \xrightarrow[x\to\pm\infty]{} \boxed{-\infty}$$

$$\Rightarrow \quad \nexists \lim_{(x,y)\to\infty_2} f(x,y) .$$ ■

Ex. $\lim_{(x,y)\to\infty_2} \underbrace{(x^2+y^2+xy)}_{f(x,y)}$ (∃ and value)

Sol:

$$f(x,0) = x^2 \xrightarrow[x\to\pm\infty]{} +\infty$$



$$f(x,mx) = x^2 + \underset{\geq 0}{m^2}x^2 + m x^2$$

$$= \boxed{(m^2+m+1)}\, x^2 \xrightarrow[x \to \pm\infty]{} +\infty ,$$

$$(x, mx) \to \infty_2 \iff \underbrace{\|(x, mx)\|}_{\begin{gathered}\| \\ \sqrt{x^2+m^2x^2}\end{gathered}} \to +\infty$$

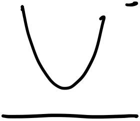
$$= \sqrt{x^2(1+m^2)} = |x| \sqrt{1+m^2}$$

$$\boxed{m^2+m+1=0}$$

$$\Updownarrow$$

$$\Delta = 1^2 - 4 \cdot 1 \cdot 1$$

$$= 1-4 = -3 < 0 \iff \boxed{|x| \to +\infty}$$



Can we conclude that

$$\exists \lim_{(x,y) \to \infty_2} f = +\infty \ ?$$

No! (sects. are never suff. to prove $\exists$)

Let's look at f in polar coords:

$$f(x,y) = \rho^2 + \rho^2 \cos\theta \,\cancel{\rho} \sin\theta$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$x^2+y^2 = \rho^2$

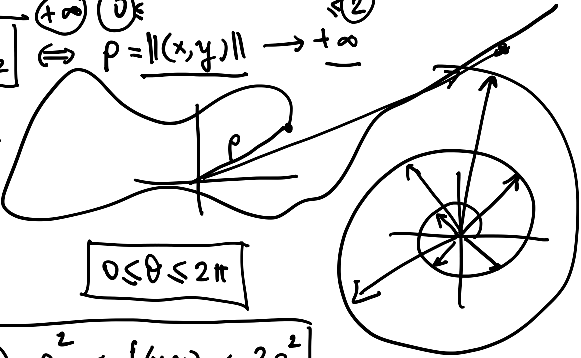
$-1 \leq \qquad \leq 1$

$$\cdot\ f(x,y) = \underbrace{\boxed{\rho^2}}_{+\infty}\ \underbrace{\left(1 + \underbrace{\cos\theta\sin\theta}\right)}_{0 \leq \qquad \leq 2} \leftarrow$$

$$\boxed{(x,y) \to \infty_2} \iff \rho = \underline{\|(x,y)\|} \to \underline{+\infty}$$

$$\Updownarrow$$

$$\boxed{\rho \to +\infty}$$



$$\boxed{0 \leq \theta \leq 2\pi}$$

$$\Rightarrow \boxed{0} = \boxed{0 \cdot \rho^2 \leq f(x,y) \leq 2\rho^2}$$

$$\begin{array}{ccccc} 0 & \leq & \underline{f(x,y)} & \leq & 2\rho^2 \\ \downarrow & & & & \downarrow \\ 0 & & & & +\infty \end{array} \qquad \rho \to +\infty$$

they are not policemen

$$f(x,y) = \rho^2\left(1 + \frac{\overset{\sin(2\theta)}{\boxed{2\cos\theta\sin\theta}}}{2}\right)$$

$$f = \rho^2\left(\textcircled{1} + \frac{1}{2}\underbrace{\sin(2\theta)}_{-1 \leq \quad \leq 1}\right)$$

$$-1/2 \leq \qquad \leq 1/2$$

$$\boxed{1/2} \leq \qquad \leq \boxed{3/2}$$

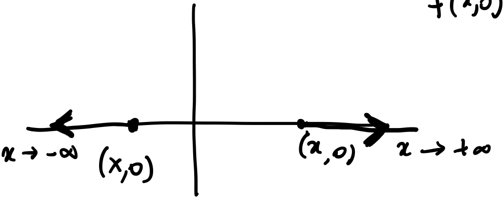
$$\Rightarrow \quad \underset{\substack{\downarrow \\ +\infty}}{\tfrac{1}{2}\rho^2} \leq \underset{\substack{\downarrow \\ +\infty}}{f(x,y)} \leq \underset{\substack{\downarrow \\ +\infty}}{\tfrac{3}{2}\rho^2}$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to\infty_2} f = +\infty . \quad \blacksquare$$

Do Ex 1.7.9 #1, 2, 3,

#1. $\displaystyle\lim_{(x,y)\to\infty_2} \left(x^3 + xy^2 - y^2\right)$

$$f(x,0) = x^3 \begin{matrix} \xrightarrow{x\to+\infty} +\infty \\ \xrightarrow{x\to-\infty} -\infty \end{matrix}$$



$$\Rightarrow \quad \nexists \lim_{(x,y)\to\infty_2} f . \quad \blacksquare$$

Do #2, #3 +

Ex: $\displaystyle\lim_{\boxed{(x,y)\to\infty_2}} \Big[\underbrace{(x^2+y^2)^2 - xy^3}\Big]$

Sol: Here $f(x,y)$

$$f(x,0) = x^4 \xrightarrow[x \to \pm\infty]{} \boxed{+\infty}$$

$$f(x,y) \underset{\text{pol. coords}}{=} (\rho^2)^2 - \rho\cos\theta\,(\rho\sin\theta)^{③}$$

$$= \rho^4 - \rho^4\cos\theta(\sin\theta)^3$$

$$= \underset{\downarrow\ +\infty}{\rho^4}\Big(\underbrace{1 - \overset{-1\le \qquad \le 1}{\boxed{\cos\theta(\sin\theta)^3}}}_{\boxed{0}\le \qquad \le \boxed{2}}\Big) \Rightarrow$$

$$(x,y) \to \infty_2 \iff \rho \to +\infty$$

$$0 \le \theta \le 2\pi$$

$$\boxed{0} \le f \le \boxed{2\rho^4}$$ not good to conclude!

$$= \rho^4\Big(1 - \frac{1}{2}\underbrace{\boxed{2\cos\theta\sin\theta}}_{\sin 2\theta}(\sin\theta)^2\Big)$$

$$= \rho^4\underbrace{\Big(1 - \frac{1}{2}\underbrace{\sin 2\theta\cdot(\sin\theta)^2}_{-1\le \qquad \le 1}\Big)}_{1/2 \le \qquad \le 3/2}$$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ +\infty}}{\boxed{\frac{1}{2}\rho^4}} \le \underset{\substack{\downarrow \\ +\infty}}{f} \le \underset{\substack{\downarrow \\ +\infty}}{\boxed{\frac{3}{2}\rho^4}}$$ good bound!

$$\Rightarrow \quad \exists \lim_{\infty_2} f(x,y) = +\infty. \qquad \blacksquare$$

Ex.

- $$\lim_{(x,y)\to\infty_2} \left(x^2+y^2+y^4-\frac{3}{2}xy\right)$$
- $$\lim_{(x,y)\to\infty_2} \left(x^2y^2(x+y)-(x^2+y^2)^3\right)$$