Monday, 20 February 2023 12:00

Ex. Discuss ∃ and value of

$$\lim_{(x,y)\to\infty_2} \underbrace{\left(x^2+y^2+y^4-\frac{3}{2}xy\right)}_{f(x,y)}$$

Sol: $f : D=\mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\underbrace{f(x,0)}_{\substack{\downarrow\\ \infty_2}} = x^2 \xrightarrow[x\to\pm\infty]{} \boxed{+\infty}$$

To discuss ∃ let's give a look to f in polar coords:

$$f(x,y) \underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \boxed{\rho^2} + (\rho\sin\theta)^4 - \frac{3}{2}\boxed{\rho^2}\cos\theta\sin\theta$$

$$\left((x,y)\to\infty_2 \iff \overset{\rho}{\overset{\|}{\boxed{\|(x,y)\|}}} \longrightarrow +\infty \iff \underset{\substack{\text{no cond on } \theta\\ \boxed{0\le\theta\le 2\pi}}}{\rho \to +\infty} \right)$$

$$= \rho^2\left(1-\frac{3}{2}\,\frac{\overset{\sin(2\theta)}{\boxed{2\cos\theta\sin\theta}}}{2}\right)+\rho^4(\sin\theta)^4$$

$$\boxed{f(x,y)} = \underset{\underset{+\infty}{\downarrow}}{\rho}^2 \underbrace{\left(1 - \frac{3}{4} \overset{-1 \leq \quad \leq 1}{\sin(2\theta)}\right)}_{\frac{1}{4} \leq \quad \leq \frac{7}{4}} \overset{+}{\boxed{+ \underset{\underset{+\infty}{\downarrow}}{\rho^4} \underbrace{(\sin\theta)^4}_{0 \leq \quad \leq 1}}}$$

$$\geq \rho^2 \underbrace{\left(1 - \frac{3}{4}\sin(2\theta)\right)}_{\geq 1/4}$$

$$f(x,y) \geq \frac{1}{4}\rho^2$$

$$\Rightarrow \quad f(x,y) \geq \frac{\rho^2}{4} \xrightarrow[\rho \to +\infty]{} +\infty$$

$$\Rightarrow \quad \exists \lim_{(x,y) \to \infty_2} f = +\infty. \quad \blacksquare$$

Ex. $\lim\limits_{(x,y) \to \infty_2} \underbrace{\left[x^2y^2(x+y) - (x^2+y^2)^3\right]}_{f(x,y)}$

Sol:

$$\underset{\underset{\infty_2}{\downarrow}}{f(x,0)} = -\underbrace{(x^2)}^3 = -x^6 \xrightarrow[x \to \pm\infty]{} \boxed{-\infty}$$

$$\|(x,0)\| \longrightarrow +\infty$$

$$\|$$

$|x|$

Is the lim $= -\infty$?

$$f(x,y) = \underbrace{\boxed{\rho^5}}_{\to +\infty} \underbrace{\underbrace{\cos^2\theta}_{\substack{+ \\ \geq 0}} \underbrace{\sin^2\theta}_{\substack{+ \\ \geq 0}} \overbrace{(\cancel{\rho}\cos\theta + \cancel{\rho}\sin\theta)}^{+/-}}$$

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$-(\rho^2)^3 = -\rho^6$$

$$f(x,y) = \rho^5 g(\theta) - \underbrace{\rho^6}_{\to +\infty}$$

$$\boxed{(x,y) \to \infty_2} \iff \underbrace{\boxed{\rho \to +\infty}}_{0 \leq \theta \leq 2\pi}$$

$$= -\rho^6 \left[1 - \underbrace{\underbrace{\frac{1}{\rho}}_{\to 0} \underbrace{g(\theta)}_{-2 \leq \quad \leq 2}}_{\to 0} \right]^{\text{bounded}} \to 1$$

$$g \in \mathcal{C}([0,2\pi]) \quad \boxed{g(\theta)} = \underbrace{\cos^2\theta}_{0 \leq \quad \leq 1} \underbrace{\sin^2\theta}_{0 \leq \quad \leq 1} (\cos\theta + \sin\theta)$$

$\Downarrow$ Weierstr.

$$m \leq g(\theta) \leq M \qquad -2 \leq g(\theta) \leq 2$$

$$\boxed{f(x,y) = \underbrace{-\rho^6}\cdot 1_{\rho,\theta}} \longrightarrow -\infty.$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to\infty_2} f = -\infty. \quad \blacksquare$$

Ex 1.7.8

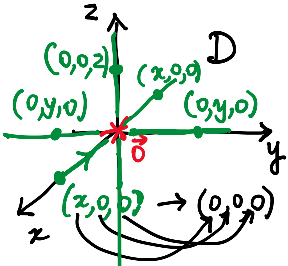
$$\# \quad \lim_{(x,y,z)\to\vec{0}} \frac{xyz}{x^2+y^2+z^2}.$$

Sol: Here $f = f(x,y,z) : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \neq 0\}$$

$$\Big(\underset{+}{x^2} + \underset{+}{y^2} + \underset{+}{z^2} = 0 \iff x^2 = y^2 = z^2 = 0 \iff x = y = z = 0\Big)$$

$$\Rightarrow \quad D = \mathbb{R}^3 \setminus \{\vec{0}\}$$



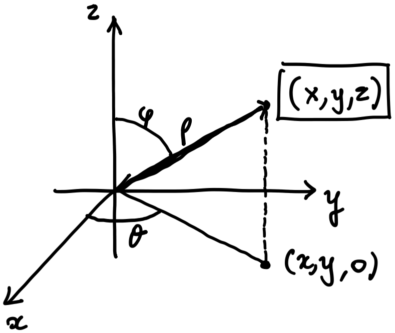
Sections:

$$f(x,0,0) = \frac{x\cdot 0\cdot 0}{x^2+0^2+0^2} = \frac{0}{x^2} = 0 \xrightarrow[x\to 0]{} \boxed{0.}$$

$$f(0,y,0) = 0 \xrightarrow[y\to 0]{} \boxed{0}$$

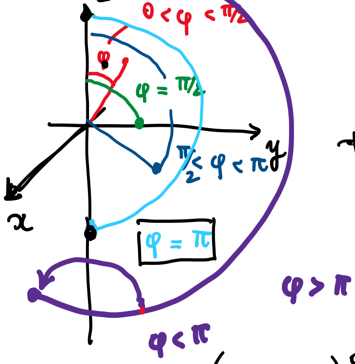
$$f(0,0,z) \equiv 0 \xrightarrow[z \to 0]{} \boxed{0}$$

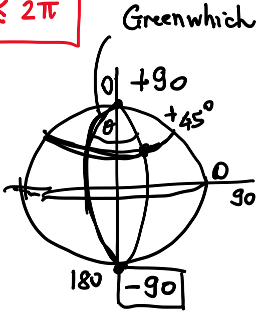
Spherical Coords



Spher. coord

$$\boxed{\rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi}$$

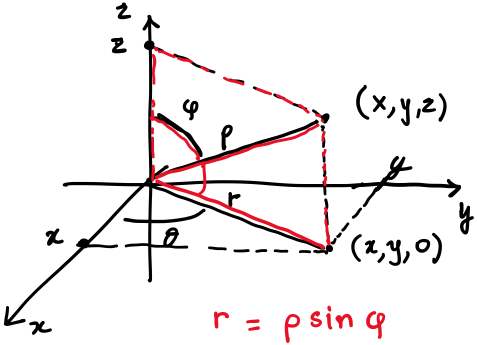




$$\text{Lat} = \left(\tfrac{\pi}{2} - \varphi\right)^\circ$$

co-latitude

$$\text{Long} = (\theta)^\circ$$



$$r = \rho \sin\varphi$$

$$\begin{cases} x = \boxed{r}\cos\theta \\ y = \boxed{r}\sin\theta \\ z = \rho\cos\varphi \end{cases} \Rightarrow \begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\boxed{x^2+y^2+z^2} = \|(x,y,z)\|^2 = \boxed{\rho^2}$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

Ret to the limit we have

$$\underbrace{f(x,y,z)}_{\text{spher. coords}} = \frac{\rho^{\cancel{3}}\sin^2\varphi\cos\theta\sin\theta\cos\varphi}{\cancel{\rho^2}}$$

$$= \overset{\overset{0}{\uparrow}}{\boxed{\rho}} \cdot \boxed{-1 \leq} \boxed{\sin^2\varphi \,|\, \cos\theta \,|\, \sin\theta \,|\, \cos\varphi} \boxed{\leq 1} \leftarrow$$

$$(x,y,z) \to \vec{0} \iff \|(x,y,z)\| = \boxed{\rho \to 0}$$

$$\boxed{\begin{array}{l} 0 \le \varphi \le \pi \\ 0 \le \vartheta \le 2\pi \end{array}}$$

$$\begin{tikzcd} \boxed{-\rho \le f(x,y,z) \le \rho} \arrow[d] & 0 \le |f(x,y,z)| \le \rho \\ 0 & \end{tikzcd}$$

$$\Rightarrow \quad \exists \lim_{(x,y,z)\to\vec{0}} f = 0$$ by the Police thm ■

Ex 1.2.12

Compute $$\lim_{(x,y,z)\to\infty_3} \underbrace{\left[(x^2+y^2+z^2)^2 - xyz\right]}_{f(x,y,z)}$$

Sol: Here

$$\begin{tikzcd} f(x,0,0) \arrow[d] & = (x^2)^2 = x^4 \arrow[r] & \boxed{+\infty} \\ \infty_3 \arrow[r, Leftrightarrow] & x \to \pm\infty & \end{tikzcd}$$

Let's give a look to f in sph. coords

$$f(x,y,z) \underset{\text{sph. coords}}{=} (\rho^2)^2 - \underbrace{\rho}_{}\sin\varphi\cos\theta\,\underbrace{\rho}_{}\sin\varphi\sin\theta\,\underbrace{\rho}_{}\cos\varphi$$

$$f = \underbrace{\overset{+\infty}{\boxed{\rho^4}} \ominus \overset{+\infty}{\boxed{\rho^3}}}_{}\ \overset{-1\le \qquad \le 1}{\boxed{\sin^2\varphi\,\cos\theta\sin\theta\,\cos\varphi}}$$

$$(x,y,z) \to \infty_3 \iff \rho \to +\infty$$

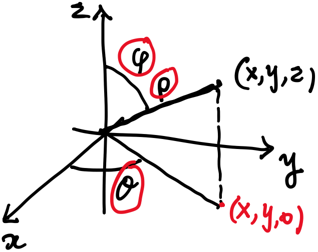
$$0 \le \varphi \le \pi$$

$$0 \le \theta \le 2\pi.$$

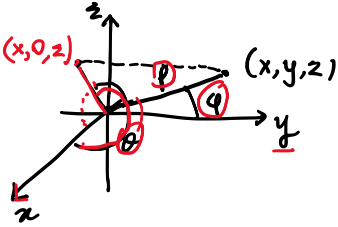
$$\begin{array}{ccccc} \rho^4-\rho^3 & \le & f & \le & \rho^4+\rho^3 \\ \downarrow & & \downarrow & & \downarrow \\ +\infty & & +\infty & & +\infty \end{array} \qquad \rho \to +\infty$$

$$\Rightarrow \quad \exists \lim_{(x,y,z)\to\infty_3} f = +\infty \quad \blacksquare$$

Rmk: We introduced sph. coords. "respect to the z-axis"



We could also choose

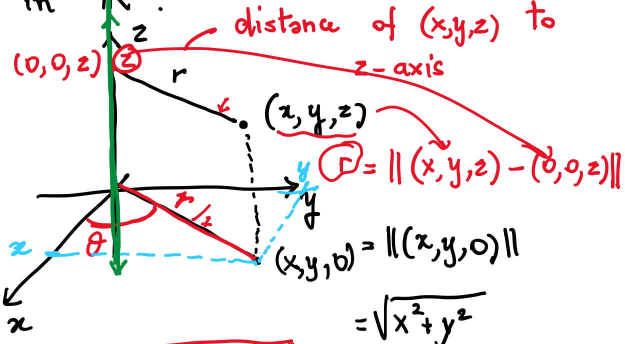


$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \cos\varphi \\ z = \rho \sin\varphi \sin\theta \end{cases}$$

∎

Cylindrical Coords

There is another important system of coords for pts in $\mathbb{R}^3$.



$r = \|(x,y,z) - (0,0,z)\|$

$(x,y,0) = \|(x,y,0)\|$

$= \sqrt{x^2+y^2}$

$$\boxed{r = \sqrt{x^2+y^2}}$$

$$(x,y,z) \longleftrightarrow \boxed{(r,\theta,z)}$$

$r \geq 0 \quad 0 \leq \theta \leq 2\pi, \quad z \in \mathbb{R}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\boxed{\overline{r^2 = x^2+y^2}}$$

$$\underbrace{x^2+y^2}_{r^2}+z^2 = \boxed{r^2+z^2}$$

Ex 1.2.13

$$\lim_{(x,y,z)\to\infty_3} \underbrace{\left[(x^2+y^2)^2 + z^2 - xy \right]}_{f(x,y,z)}$$

Sol: Sections:

$$\underset{\substack{\downarrow \\ \infty_3}}{f(x,0,0)} = x^4 \xrightarrow[x\to\pm\infty]{} \boxed{+\infty}$$

$$f(x,y,z) \underset{\text{sph. coors}}{=} \rho^4\Big(\cancel{\rho^2}\sin^2\varphi\,\overbrace{(\cos^2\theta + \cancel{\rho^2}\cancel{\sin^2\varphi}\sin^2\theta)}^{=1}\Big)^2 + \rho^2\cos^2\varphi - \rho^2\sin^2\varphi\,\frac{2\cos\theta\sin\theta}{2}$$

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$$\boxed{(x,y,z)\to\infty_3} \Leftrightarrow \boxed{\rho\to+\infty}$$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$$= \boxed{\underset{\substack{\downarrow\\+\infty}}{\rho^4}\,\underset{\ge 0}{\sin^4\varphi} + \underset{\substack{\downarrow\\+\infty}}{\rho^2}\,\underset{\ge 0}{\cos^2\varphi} - \frac{1}{2}\rho^2\sin^2\varphi\,\sin(2\theta)}\ ?$$

$$\ge \rho^4/\rho^2$$

$$f(x,y,z) \underset{\text{cyl coords}}{=} (r^2)^2 + z^2 - r^2\cos\theta\sin\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$z = \boxed{z}$

$$= \boxed{r^4 + z^2 - \frac{r^2}{2}\sin(2\theta)}$$

$$\boxed{(x,y,z) \to \infty_3} \iff \|(x,y,z)\| \to +\infty$$

$$\Updownarrow$$

$$\sqrt{\underbrace{x^2+y^2}_{r^2}+z^2} \to +\infty$$

$$\iff \sqrt{r^2+z^2} \to +\infty$$

$$\iff \|(r,z)\| \to +\infty$$

$$\iff \boxed{(r,z) \to \infty_2}$$

$$\lim_{(x,y,z)\to\infty_3}$$

$$\|$$

$$\lim_{(r,z)\to\infty_2}$$

$$f(x,y,z) = r^4 + z^2 - \frac{r^2}{2}\underbrace{\sin 2\theta}_{\boxed{-1} \leq \quad \leq 1}$$

$$\boxed{r^4 + z^2 - \frac{r^2}{2} \leq f} \left(\leq r^4 + z^2 + \frac{r^2}{2} \right)$$ unnecessary

$\boxed{+\infty}$ IF $\Uparrow$ $\boxed{+\infty}$ IF $\downarrow$ $+\infty$ $(r,z)\to\infty_2$

by Police thm.

Goal: $$\boxed{\lim_{(r,z)\to\infty_2}\left(r^4+z^2-\frac{r^2}{2}\right)=+\infty}$$

$$\lim_{(r,z)\to\infty_2}\underbrace{\left(r^4+z^2-\frac{r^2}{2}\right)}_{g(r,z)}$$

$\Updownarrow$

$\|(r,z)\|\to+\infty$

$$g(r,z)=\boxed{\rho^4\cos^4\theta+\rho^2\sin^2\theta-\frac{\rho^2}{2}\cos^2\theta}$$

$r=\rho\cos\theta$

$z=\rho\sin\theta$

it looks like f in sph. coords !

$$\cancel{r^4}+z^2-\frac{r^2}{2}$$

r^2

$\boxed{(r,z)\to\infty_2}$

$$\boxed{\boxed{r^2+z^2}-\frac{r^2}{2}}=\frac{r^2}{2}+z^2\geq\frac{r^2}{2}+\frac{z^2}{2}=\frac{\boxed{r^2+z^2}}{2}=\frac{\rho^2}{2}\to+\infty$$

$z^2\geq z^2/2$

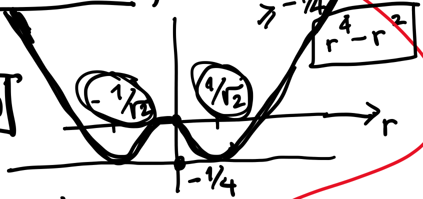
$\downarrow$ $+\infty$

$r^4 \geq r^2 \Leftrightarrow$ $\qquad \rho = \|(r,z)\| \to \boxed{+\infty}$

$r^2 \geq 1$

$$\boxed{r^4 + z^2 - \frac{r^2}{2}} = \overset{①}{\boxed{r^4 - r^2}} + \overset{②}{\boxed{r^2 + z^2 - \frac{r^2}{2}}} \to \boxed{+\infty}$$

$\boxed{+\infty}$



$r = 0, \boxed{\pm\frac{1}{\sqrt{2}}}$

$$(r^4 - r^2)' = 4r^3 - 2r = 2r(2r^2 - 1) = 0$$

$$\boxed{r^4 - r^2} \geq \boxed{\frac{1}{4}} - \boxed{\frac{1}{2}} = \boxed{-\frac{1}{4}}$$

$$\geq \underbrace{\boxed{-\frac{1}{4}} + \underbrace{r^2 + z^2 - \frac{r^2}{2}}_{\to +\infty}}_{\to +\infty}$$

Do

1.7.8 #3

1.7.9 #6

#7

#5

Before do

$$\lim_{(x,y,z)\to\infty_3} \left(x^2 + y^2 + z^2 - xy\right)$$

$$\lim_{(x,y,z)\to\infty_3} \left((x^2+y^2)^2 + (x^2+z^2)^2 - xyz\right)$$