Tuesday, 21 February 2023 08:44

Continuity

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$

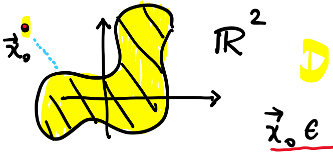
Def: We say that $\vec{F}$ is cont at $\vec{x}_0$

$(\vec{x}_0 \in D \cap \mathrm{Acc}\, D)$ if

$$\lim_{\vec{x} \to \vec{x}_0} \vec{F}(\vec{x}) = \vec{F}(\vec{x}_0) \quad \leftarrow$$

Rmk: In gen it may happens that

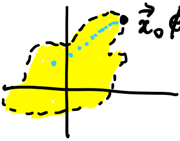
- $\vec{x}_0 \in D$ but $\vec{x}_0 \in \mathrm{Acc}\, D$



$\vec{x}_0 \in D$ but

$\vec{x}_0 \notin \mathrm{Acc}\, D$

- $\vec{x}_0 \in \mathrm{Acc}\, D$ but $\vec{x}_0 \in D$



$\vec{x}_0 \notin D$

$\vec{x}_0 \in \mathrm{Acc}\ D$

Examples:

- $p(\vec{x})$ polynomial

$$\left(p(x) = \sum_{j=0}^{k} c_j x^j \qquad x \in \mathbb{R} \right)$$

$$p(\vec{x}) = \underbrace{\sum}_{\text{finite sum of}} c_j \, x_1^{j_1} x_2^{j_2} \cdots x_d^{j_d}$$

$$p(x,y) := xy^2 + 2x + y$$

$$p(x,y,z) := 3x^2 + zy + 4x^3y^2 + z^3$$

Every polynomial is cont. at $\forall \vec{x}_0 \in \mathbb{R}^d$

- $\varphi(p(\vec{x}))$ where $\varphi = \varphi(t)$ $t \in \mathbb{R}$

If $\varphi \in \mathcal{C}(D) \Rightarrow \varphi \circ p \in \mathcal{C}$ where defined

$$f(x,y) = \log(\underbrace{1 - x^2 - y^2})$$

$$= \varphi(p(x,y)) \quad \text{where}$$

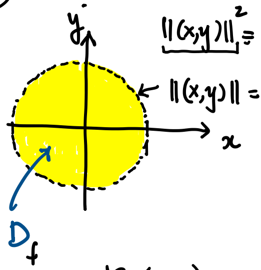
$$p(x,y) = 1 - x^2 - y^2$$

$$\varphi(t) = \log t \in \mathcal{C}(]0,+\infty[)$$

$f = \varphi \circ p \in \mathcal{C}$ on its domain

$$D_f = \{ (x,y) \in \mathbb{R}^2 \; ; \; 1 - x^2 - y^2 > 0 \}$$

$$\underbrace{\|(x,y)\|^2}_{} = x^2 + y^2 < 1$$



$\Rightarrow \quad f \in \mathcal{C}(D_f)$. ■

- $f(x,y) = \sqrt{\underbrace{1 - x + y}_{}} = \varphi(p(x,y))$

$p(x,y) = 1 - x + y$

$\varphi(t) = \sqrt{t} \in \mathcal{C}([0,+\infty[)$

$\Rightarrow \quad f \in \mathcal{C}(D_f)$

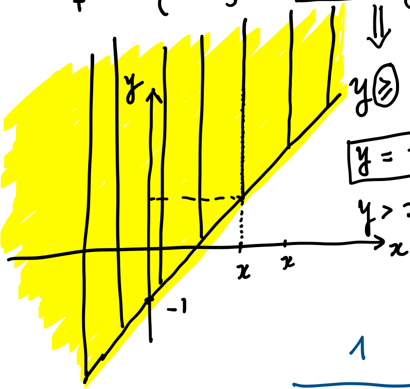
$$D_f = \{ (x,y) : \boxed{1 - x + y \geq 0} \}$$

$\Downarrow$

$y \geq x - 1$

$\boxed{y = x - 1}$

$y > x - 1$



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- $f(x,y,z) := \sqrt{\underbrace{9-y^2-x^2}_{}}$

Here $\quad f = \varphi(p(x,y,z)) \quad$ where

$$p(x,y,z) = 9-y^2-x^2$$

$$\varphi(t) = \frac{1}{\sqrt{t}} \in \mathcal{C}(]0,+\infty[)$$

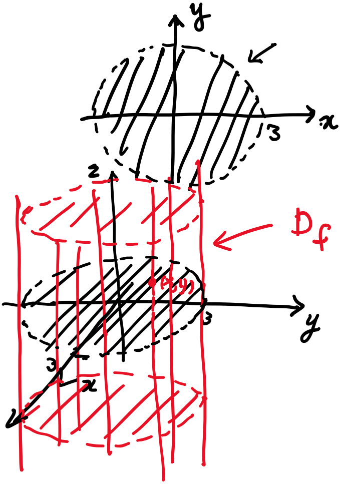
$$\Rightarrow f \in \mathcal{C}(D_f)$$

$$D_f = \{ \underline{(x,y,z)} \in \mathbb{R}^3 : \underbrace{9-y^2-x^2>0}_{} \}$$

$$\longrightarrow \quad y^2+x^2 < 9$$

$$\shortparallel$$

$$\|(x,y)\|^2 < 9 \iff \|(x,y)\| < 3$$



General rules:

alg operations on cont functs leads to cont functs where they are dfd.

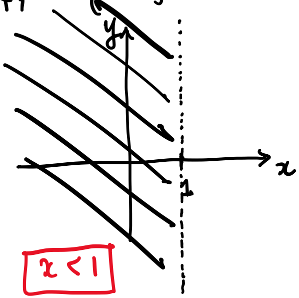
$$f(\underline{x,y}) = \underset{\underline{f_1(x,y)}}{\boxed{\log(1-x)}} + \underset{\underline{f_2(x,y)}}{\boxed{\sqrt{1-xy}}}$$

$$f_1(x,y) = \varphi(p(x,y))$$

$$\varphi(t) = \log t \qquad f_1 \in \mathcal{C}(D_{f_1})$$

$$p(x,y) = 1-x$$

$$D_{f_1} = \{(x,y) : 1-x>0\} = \{(x,y) : \underline{x<1}\}$$



$$f_2(x,y) = \sqrt{1-xy} = \varphi(p(x,y))$$

where

$$\varphi(t) = \sqrt{t}, \quad p(x,y) = 1-xy$$

$$\Rightarrow \quad f_2 \in \mathcal{C}(D_{f_2})$$

$$D_{f_2} = \{(x,y) : 1 - xy \geqslant 0\}$$

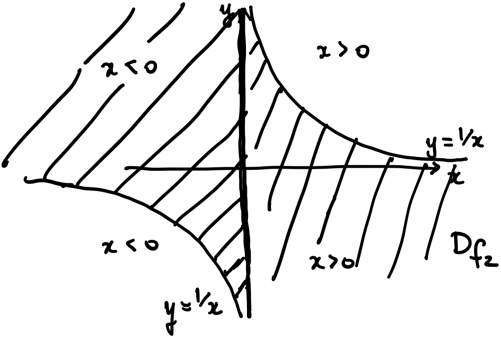
$$\Updownarrow$$

$$\boxed{xy \leqslant 1}$$

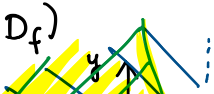
$$\Updownarrow$$

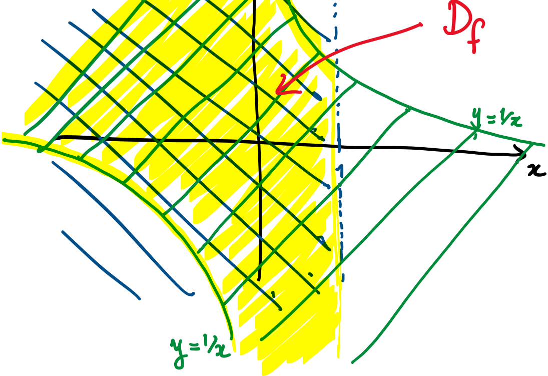
$$\underline{\cancel{y \leqslant 1/x}} \quad ?$$

$$xy \leqslant 1 \iff \left\{\begin{array}{lll} \underline{x > 0} & y \leqslant 1/x & \\ \boxed{x = 0} & 0 \leqslant 1 & \underline{\text{always true}} \\ x < 0 & y \geqslant 1/x & \end{array}\right.$$



So $f \in \mathscr{C}(D_f)$





Prop: Let $\underset{\uparrow}{\vec{F}(\vec{x})} = \underbrace{(f_1(\vec{x}), f_2(\vec{x}), \qquad , f_m(\vec{x}))}_{\text{components of } \vec{F}}$

(each component $f_j = f_j(\vec{x}) \in \mathbb{R}$).

$\vec{F}$ is cont at $\vec{x}_0 \iff \forall f_j$ is cont at $\vec{x}_0$

(continuity of a vector valued funct can be checked component wise)

Ex: $\vec{F} = \vec{F}(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\vec{F}\underbrace{(x,y)}_{\mathbb{R}^2} = (\overset{f_1(x,y)}{xy}, \overset{f_2(x,y)}{x+y}, \overset{f_3(x,y)}{x^2+y^2})$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

According to the Prop, $\vec{F}$ is cont at $(x_0,y_0) \iff$ each f_j is cont at (x_0,y_0).

Since f_j is a polyn. $\forall j=1,2,3$

$$\forall f_j \in \mathcal{C}(\mathbb{R}^2) \Longrightarrow \vec{F} \in \mathcal{C}(\mathbb{R}^2). \quad \blacksquare$$

- A gen remarkable example is

multiplication lines by cols.

$$\vec{F}(\vec{x}) = \underset{\substack{\uparrow \\ \text{matrix}}}{A}\,\vec{x} = ⓜ\left\{\overset{d}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{md} \end{bmatrix}}\right. \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

$$\vec{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^{ⓜ}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{md}x_d \end{pmatrix} \left.\vphantom{\begin{pmatrix}a\\a\\a\\a\end{pmatrix}}\right\} ⓜ \text{ components}$$

(A must be an $\boxed{m \times d}$ matrix)

$$\vec{F} \in \mathcal{C}(\mathbb{R}^d).$$

Indeed, each of the components

$$f_j(x_1 \text{—} x_d) = \underbrace{a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jd}x_d}_{\text{polynomial}}$$

$$\Rightarrow \forall f_j \in \mathcal{C}(\mathbb{R}^d)$$

$$\Rightarrow \vec{F} \in \mathcal{C}(\mathbb{R}^d). \quad \blacksquare$$

- Norm. $f(\vec{x}) := \|\vec{x}\|$

$$f : \mathbb{R}^d \longrightarrow \mathbb{R}$$

$f \in \mathcal{C}(\mathbb{R}^d)$.

Indeed

$$f(\vec{x}) = \sqrt{\underbrace{x_1^2 + x_2^2 + \dots + x_d^2}_{\text{polynomial}}}$$

$\varphi(t) = \sqrt{t}$

$$\Rightarrow f \in \mathcal{C}(D_f) = \mathcal{C}(\mathbb{R}^d) . \qquad \blacksquare$$

Properties of Cont. Functs

Refresh on few known and important facts on cont funct of one real var.

Weierstass thm

If $f = f(x) : \underbrace{\boxed{[a,b]}}_{\text{closed \& bounded interval}} \longrightarrow \mathbb{R}$, $f \in \mathcal{C}([a,b])$

$\Rightarrow \boxed{\exists}\, x_{min}, x_{max} \in [a,b]$:

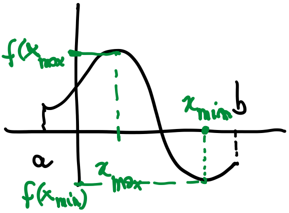
$$f(x_{min}) \leqslant f(x) \leqslant f(x_{max}) \quad \forall x \in [a,b]$$

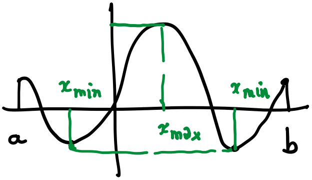
x_{min} is called global min point for f on $[a,b]$

x_{max} is called global max pt " " "

$f(x_{min})$ " " global min value " " "

$f(x_{max})$ " " global max value " " "





Pb: What if we have $f = f(x_1, \ldots, x_d)$?

Basic Topological concept

Def: Given $\vec{x}_0 \in \mathbb{R}^d$ we call

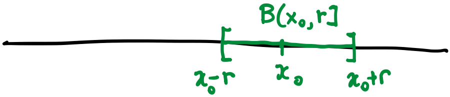
. closed ball centred at $\vec{x}_0$, radius $r > 0$, the set

$$B(\vec{x}_0, r] := \{ \vec{x} \in \mathbb{R}^d : \| \vec{x} - \vec{x}_0 \| \leq r \}$$

Examples:

$d = 1$ $\quad \mathbb{R}^d = \mathbb{R}^1 \equiv \mathbb{R}$

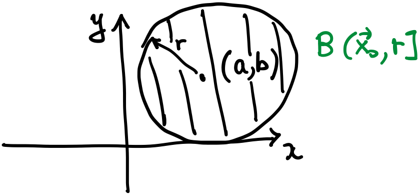
$$B(x_0, r] = \{ x \in \mathbb{R} : |x - x_0| \leq r \}$$



$d = 2, \quad \mathbb{R}^2, \quad \vec{x}_0 = (a,b)$

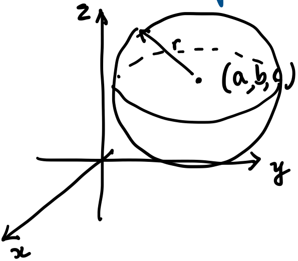
$$B(\vec{x}_0, r] = \{ (x,y) \in \mathbb{R}^2 : \underset{\substack{\| \\ = \|(x-a,\, y-b)\|}}{\|(x,y) - (a,b)\|} \leq r \}$$

$$\sqrt{(x-a)^2 + (y-b)^2} \leq r$$



$d = 3 \qquad B(\vec{x}_0, r] = \{ (x,y,z) \in \mathbb{R}^3 : \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \leq r \}$

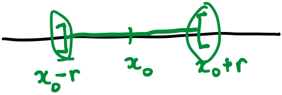
$\vec{x}_0 = (a,b,c)$



. open ball centred at $\vec{x}_0$, radius r,

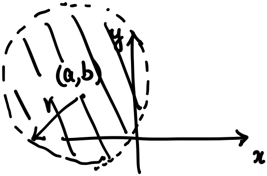
$$B(\vec{x}_0, r\boxed{[} := \{ \vec{x} \in \mathbb{R}^d : \|\vec{x} - \vec{x}_0\| < r \} \leftarrow$$

Ex: $d = 1 \qquad B(x_0, r[=]x_0 - r, x_0 + r[$



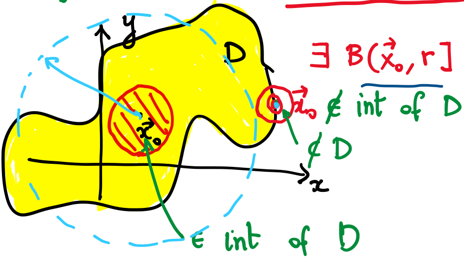
$d = 2$ $\quad B(\vec{x}_0, r[\; = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{(x-a)^2 + (y-b)^2} < r \right\}$

$\vec{x}_0 = (a,b)$



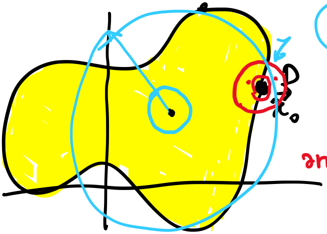
Def: Let $\vec{x}_0 \in \mathbb{R}^d$, $D \subset \mathbb{R}^d$. We say that

- $\vec{x}_0$ is in the interior of D if $\exists \, B(\vec{x}_0, r] \subset D$. ←



The set of pts that are in the int. of D is denoted by Int D (sometimes you may find the not $\mathring{D}$)

- $\vec{x}_0$ is a boundary/fronteer pt for D if



$\forall B(\vec{x}_0, r]$,

$B(\vec{x}_0, r] \cap D \neq \phi$

and $B(\vec{x}_0, r] \cap D^c \neq \phi$ ←

The set of boundary/fr. pts is denoted by $Fr(D) \equiv \partial D$.

Def: Let $D \subset \mathbb{R}^d$. We say that D is open if

$$\text{Int } D = D$$

Rmk: Notice that $\exists$, in gen,

$$\text{Int } D \subset D$$

Indeed, $\vec{x}_0 \in \text{Int } D \Rightarrow \exists B(\vec{x}_0, r] \subset \underset{\vec{x}_0}{\underset{\cup}{D}}$

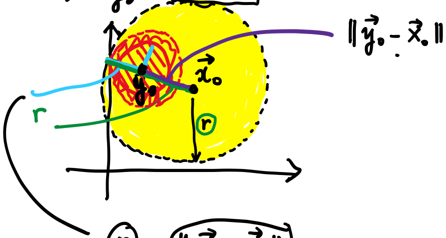
$$\Rightarrow \boxed{\text{Int } D \subset D}.$$

Example: Every open ball $\underbrace{B(\vec{x}_0, r[}_{D}$ is an open set.

Sol: What have we to check?

We have to show that $\underline{\forall \vec{y}_0} \in D = B(\vec{x}_0, r[$

$\Rightarrow \vec{y}_0 \in \boxed{\text{Int } D}.$



$r - \|\vec{y}_0 - \vec{x}_0\|$

Claim: $\underline{B(\vec{y}_0, r - \|\vec{y}_0 - \vec{x}_0\|[} \subset D = B(\vec{x}_0, r]$

Let $\vec{y} \in B(\vec{y}_0, r - \|\vec{y}_0 - \vec{x}_0\|[, \iff \boxed{\|\vec{y} - \vec{y}_0\|} < \underbrace{r - \|\vec{y}_0 - \vec{x}_0\|}$

$\underline{\vec{y} \in B(\vec{x}_0, r]} \iff \boxed{\|\vec{y} - \vec{x}_0\| < r}$

$$\boxed{\|\vec{y} - \vec{x}_0\|} = \|(\vec{y} - \vec{y}_0) + (\vec{y}_0 - \vec{x}_0)\|$$

$$\overset{\Delta}{\leq} \underbrace{\|\vec{y} - \vec{y}_0\|}_{\wedge} + \|\vec{y}_0 - \vec{x}_0\|$$

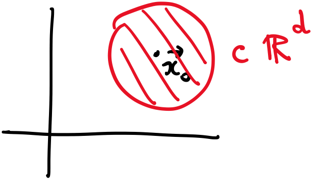
$$< (r - \cancel{\|\vec{y}_0 - \vec{x}_0\|}) + \cancel{\|\vec{y}_0 - \vec{x}_0\|}$$

$$\Rightarrow \|\vec{y} - \vec{x}_0\| < r$$

■

Rmk: For convenience we assume (by def) that $\emptyset$ (empty set) is open.

Rmk: $\mathbb{R}^d$ is clearly open.



Some good exercises on the def of open set:

Ex. Prove that:

- if A, B are open sets then $A \cup B$, $A \cap B$ are both open.
- $\underset{\neq \emptyset}{A}$ is open $\overset{\Leftarrow}{\overset{\Rightarrow}{\underset{?}{\Leftrightarrow}}}$ $\boxed{\mathrm{Fr}\, A \subset A^c}$.

(is it true or false)

Def Let $D \subset \mathbb{R}^d$. We say that D is closed iff D^c is open.

Rmk: ϕ and $\mathbb{R}^d$ are closed

(so, in part, ϕ and $\mathbb{R}^d$ are both open and closed)

Indeed

ϕ closed $\Leftrightarrow$ ϕ^c is open

$\Leftrightarrow$ $\mathbb{R}^d$ is open (true)

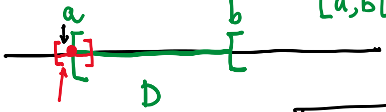
$\mathbb{R}^d$ closed $\Leftrightarrow$ $(\mathbb{R}^d)^c$ is open

$\Leftrightarrow$ ϕ " " (by def)

Rmk: We may have sets $(\phi, \mathbb{R}^d)$ which are both open & closed.

Rmk: We have also sets which are not open neither closed.

. $d=1$ $[a,b[=: D$



$\forall r \; B(a,r] \not\subset D \Rightarrow \boxed{a \notin \text{Int } D}$ $\underset{D}{\overset{a}{\Uparrow}}$

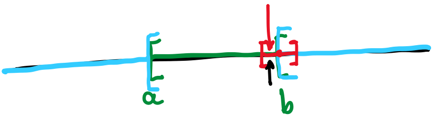
$\boxed{D \underset{\neq}{\supseteq} \text{Int } D}$

$\Rightarrow D$ is not open.

D is not closed too !

Indeed : D closed $\Leftrightarrow$ D^c open

FALSE



$D^c =]-\infty, a[\cup [b, +\infty[$

D^c is not open because,

since $\forall$ $B(b,r] \not\subset \underline{D^c}$

$\Rightarrow b \in D^c$ but $b \notin \text{Int } D^c$

$\Rightarrow \underline{\text{Int } D^c} \underset{\neq}{\subseteq} \underline{D^c}$