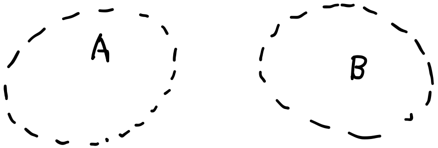
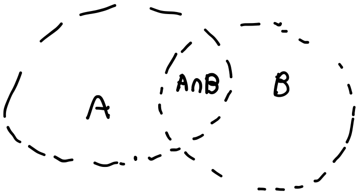
Wednesday, 22 February 2023 08:21

Ex. If A, B are open $\Rightarrow A \cap B$ is open.

Sol:



If $A \cap B = \phi$, since $\boxed{\phi \text{ is open}}$ $\Rightarrow A \cap B$ is open



If $A \cap B \neq \phi$ then we have to show that

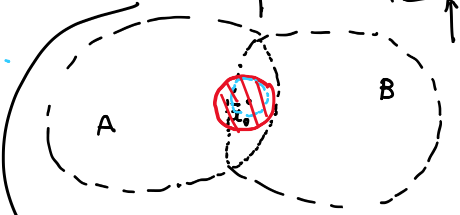
$$\mathrm{Int}(A \cap B) = A \cap B$$

$\subseteq$ always true

$\supseteq$?

$\supseteq$ means that we have to show that

$$\forall\ \boxed{\vec{x}_0 \in A\cap B} \Rightarrow \exists\ B(\vec{x}_0, r] \subset A\cap B.$$



$$\underline{\vec{x}_0 \in A},\ \underline{A \text{ open}} \Rightarrow \vec{x}_0 \in \overset{\overset{A}{\parallel}}{\operatorname{Int} A}$$

$$\Rightarrow\ \exists\ B(\vec{x}_0, \boxed{r_A}] \subset A$$

$$\vec{x}_0 \in B,\ B \text{ open} \Rightarrow \vec{x}_0 \in \operatorname{Int} B = B$$

$$\Rightarrow\ \exists\ B(\vec{x}_0, \boxed{r_B}] \subset B$$

Let $r := \min\{r_A, r_B\} \leq r_A, r_B$

$$\Rightarrow\ \boxed{B(\vec{x}_0, r]} \subset B(\vec{x}_0, r_A]\ \boxed{\subset A}$$

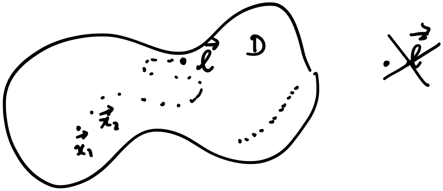
$$\subset B(\vec{x}_0, r_B]\ \boxed{\subset B}$$

$$\Rightarrow\ B(\vec{x}_0, r] \subset A\cap B$$ thesis ■

Thm (Cantor)

A set $\boxed{D \text{ is closed}}$ $\boxed{\Longleftrightarrow}$ the following property holds

$$\boxed{\forall (\vec{x}_n) \subset D :\ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \Rightarrow \vec{\ell} \in D}\ (P)$$



Proof:

Hp: $\boxed{D \quad \text{closed}}$

$\Rightarrow$

Th: (P) holds

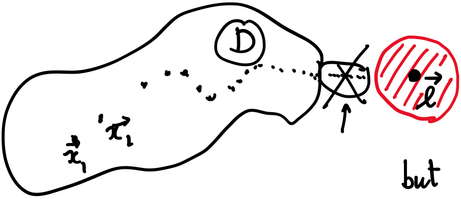
To prove that (P) is true we have to show that

IF $\boxed{(\vec{x}_n) \subset D}$ is such that

$$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$$

THEN $\boxed{\vec{\ell} \in D}$

By contradiction: if $\boxed{\vec{\ell} \notin D} \iff \vec{\ell} \in D^c$



$\Downarrow$ open

$\exists B(\vec{\ell}, r) \subset D^c$

but this is impossible!

Indeed, since $\vec{x}_n \longrightarrow \vec{\ell} \iff \|\vec{x}_n - \vec{\ell}\| \to 0$

$\Rightarrow \|\vec{x}_n - \vec{\ell}\| \le r$ for $n \geq N$

$$\Rightarrow \quad \vec{x}_n \in B(\vec{\ell}, r] \quad \forall n \geq N$$
$$\subset \textcircled{D^c}$$

$$\Rightarrow \quad \vec{x}_n \notin D \quad \forall n \geq N \quad !$$

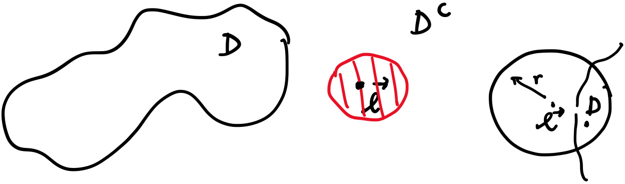
$$\Rightarrow \quad \vec{\ell} \in D.$$

$\Leftarrow$ Hp: $\boxed{\text{(P) true}}$

Th: D closed $\Leftrightarrow$ $\boxed{D^c \text{ open}}$

Pick $\boxed{\vec{\ell} \in D^c}$. Goal is to show

$$\boxed{\exists\, B(\vec{\ell}, r] \subset D^c.}$$



By contradiction: assume

$$\boxed{\nexists\, B(\vec{\ell}, r] \subset \underline{D^c}}$$

$$\Leftrightarrow \quad \forall\, \cancel{B(\vec{\ell}, r] \subset D}$$

$$\Leftrightarrow \quad \forall\, B(\vec{\ell}, \underset{\uparrow}{\textcircled{r}}] \cap D \neq \emptyset$$

$\boxed{r = 1/n}$, $n = 1, 2, 3, \ldots$

$$B(\vec{\ell}, 1/n] \cap \underset{\smile}{D} \neq \emptyset \Rightarrow \exists\, \textcircled{$\vec{x}_n$} \in \underset{\smile}{D} \cap B(\vec{\ell}, 1/n]$$

$$\Downarrow$$

$$(\vec{x}_n) \subset D, \quad \boxed{\|\vec{x}_n - \vec{\ell}\| \le \textcircled{\tfrac{1}{n}} \xrightarrow[n\to+\infty]{} 0}$$

$$(\vec{x}_n) \subset D : \quad \underline{\vec{x}_n \to \vec{\ell} \in \mathbb{R}^d} \underset{(P)}{\Rightarrow} \boxed{\vec{\ell} \in D}$$

imposs!

$$\Rightarrow \quad \exists\ B(\vec{\ell}, r] \subset D^c \Rightarrow \vec{\ell} \in \operatorname{Int} D^c$$

$$\Rightarrow D^c \subset \operatorname{Int} D^c$$

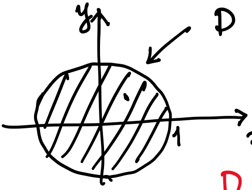
$$\Rightarrow \boxed{D^c = \operatorname{Int} D^c}$$

$$\Rightarrow D^c \text{ open} \ \blacksquare$$

From Cantor characterization of closed sets we get an easy test to check if a "concrete" set is open or closed. Let's see how.

A gen way to def a set in $\mathbb{R}^d$ is through a number of inequalities and/or equalities.

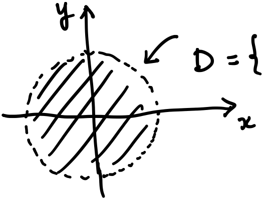
Ex:



$$D = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| \le 1\}$$

$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \le 1\}$$

$$D = \{(x,y) \in \mathbb{R}^2 : \boxed{x^2+y^2 \le 1}\}$$

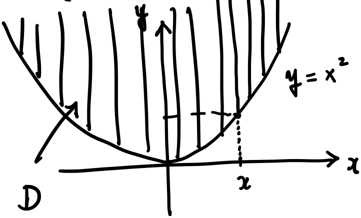


$$D = \{(x,y)\in\mathbb{R}^2 : \boxed{x^2+y^2<1}\}$$

large/weak inequality

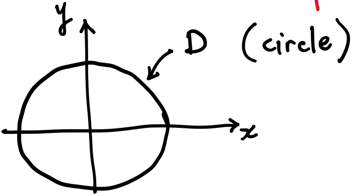
strict/strong ineq.

$$D = \{(x,y)\in\mathbb{R}^2 : \boxed{y \geq x^2}\}$$



$$D = \{(x,y)\in\mathbb{R}^2 : \boxed{x^2+y^2=1}\}$$

equality

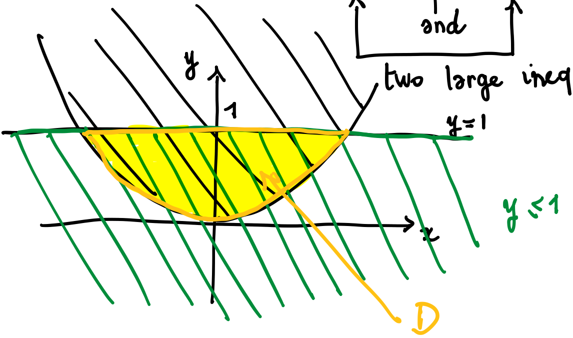


$$D = \{(x,y)\in\mathbb{R}^2 : x^2 \leq y \leq 1\}$$

$= \{ \underline{(x,y)} \in \mathbb{R}^2 : \boxed{y \geq x^2}, \; \underline{y \leq 1} \}$

and

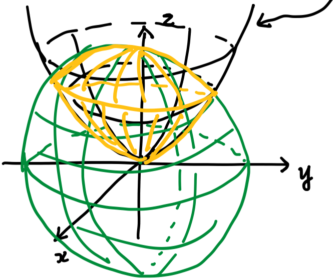
two large ineq.



$D = \{ (x,y,z) \in \mathbb{R}^3 : z \geq x^2+y^2, \; x^2+y^2+z^2 \leq 1 \}$

two weak ineq.

$\sqrt{x^2+y^2}$



In general, we may define a set in one of the following forms:

$\{ \vec{x} \in \mathbb{R}^d : f_1(\vec{x}) \geq 0, \; f_2(\vec{x}) \geq 0, \dots, \; f_m(\vec{x}) \geq 0 \}$

(set defd through large inequalities)

$\{ \vec{x} \in \mathbb{R}^d : f_1(\vec{x}) > 0, \; f_2(\vec{x}) > 0, \dots, \; f_m(\vec{x}) > 0 \}$

(set defd through strict ineq)

$$\{\vec{x}\in\mathbb{R}^d : f_1(\vec{x})=0,\ f_2(\vec{x})=0, \dots, f_m(\vec{x})=0\}$$

(set defd through equalities)

Functs $f_1, f_2, \dots f_m : \mathbb{R}^d \longrightarrow \mathbb{R}$

are called constraints

Corollary (of Cantor thm)

If constraints $\underline{f_1, f_2, \dots f_m \in \mathscr{C}}$ (cont functs)

then

- $\{f_1 \geq 0, f_2 \geq 0, \dots, f_m \geq 0\}$ $\boxed{\text{is closed}}$
- $\{f_1 = 0, f_2 = 0, \dots, f_m = 0\}$ $\boxed{\text{is closed}}$
- $\{f_1 > 0, f_2 > 0, \dots, f_m > 0\}$ $\boxed{\text{is open}}$

Proof: For simplicity we will limit to the case

$$D = \{f \geq 0\},\quad D = \{f=0\},\ D = \{f>0\}$$

where $f \in \underline{\mathscr{C}(\mathbb{R}^d)}$.

Case $D = \{f \geq 0\}$ (same arg for $\underline{\{f=0\}}$)

Th: $\{f \geqslant 0\}$ is closed.

To prove this we verify that Cantor property holds:

$$\forall (\vec{x}_n) \subset D : \vec{x}_n \to \vec{\ell} \in \mathbb{R}^d \Rightarrow \vec{\ell} \in D.$$

We prove this for the set

$$D = \{f(\vec{x}) \geqslant 0\}.$$

If $(\vec{x}_n) \subset D = \{f \geqslant 0\}$ $\left(\Leftrightarrow \ f(\vec{x}_n) \geqslant 0 \ \forall n\right)$

is s.t. $\underline{\vec{x}_n \to \vec{\ell} \in \mathbb{R}^d}$ $\Rightarrow$ $\boxed{\vec{\ell} \in D}$

$$\boxed{f(\vec{x}_n) \,\big|\, \geqslant 0} \quad \forall n \in \mathbb{N}$$

$\boxed{f \in \mathcal{C}}$

$$\downarrow \vec{\ell}$$

$\boxed{f(\vec{\ell}) \,\big|\, \geqslant 0}$ because of the permanence of sign

$\underline{\text{Case}}$: $D = \{f > 0\}$. We aim to show

D is open $\Leftrightarrow$ D^c closed

$D = \left(D^c\right)^c$, $D^c = \{f \leqslant 0\}$ closed, $f \in \mathcal{C}$ ∎

Example

- $D = \{ (x,y) : x^2 \leq y \leq 1 \}$

$$= \{(x,y) : y \geq x^2 ,\ y \leq 1\}$$

$$= \{(x,y) : \underset{\underset{f_1(x,y)}{\parallel}}{y - x^2} \geq 0 ,\ \underset{\underset{f_2(x,y)}{\parallel}}{1 - y} \geq 0\}$$

$$= \{(x,y) : f_1 \geq 0 ,\ f_2 \geq 0\}$$

$f_1 , f_2 \in \mathcal{C}(\mathbb{R}^2)$ (they are polyn.)

$\Rightarrow$ D is closed. ■

- $D = \{(x,y,z) \in \mathbb{R}^2 : \underbrace{z - xy + xyz}_{\parallel} = 0\}$

Here $D = \{ f = 0 \}$, $f(x,y,z) \in \mathcal{C}(\mathbb{R}^3)$

$\Rightarrow$ D is closed. ■

Def: We say that $D \subset \mathbb{R}^d$ is **bounded**

if $\boxed{\exists \boxed{M} \in \mathbb{R} : \underline{\|\vec{x}\| \leq M} \quad \forall \vec{x} \in D.}$

Ex.1 $D = \{(x,y) : \underline{x^2 + y^2 \leq 4}\}$

This D is closed (because it is defd through a large ineq involving a cont funct $f(x,y) \geq 0$ $\quad f = 4 - x^2 - y^2$ poly.)

Is it bounded? Yes because

$$(x,y)\in D \iff x^2+y^2 \le 4$$

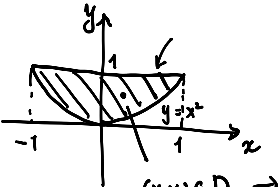
$$\iff \|(x,y)\|^2 \le 4$$

$$\iff \|(x,y)\| \le 2 =: M \quad \forall (x,y)\in D \qquad \blacksquare$$

Ex 2 $D = \{(x,y) : \underline{x^2 \le y \le 1}\}$

We already discussed about closedness.
Let's look to boundedness.

Sol 1 We draw D



$$(x,y)\in D \Rightarrow \boxed{\begin{array}{c}0\le y\le 1\\ -1\le x\le 1\end{array}} \begin{array}{l}\Rightarrow \boxed{y^2\le 1}\\ \Rightarrow \boxed{x^2\le 1}\end{array}$$

$$\Rightarrow \|(x,y)\| = \sqrt{\underbrace{x^2}_{\substack{\wedge\!\!/\\ 1}}+\underbrace{y^2}_{\le 1}} \le \sqrt{\underbrace{1+1}} = \sqrt{2} \quad \forall (x,y)\in D.$$

$\Rightarrow$ D bounded. ✓

Sol 2 $(x,y)\in D \iff 0 \le \underline{x^2} \le \underbrace{\overset{\downarrow}{y} \le \overset{\downarrow}{1}}$

$$\left[\begin{array}{l} 0 \le y \le 1 \Rightarrow \boxed{y^2 \le 1} \\ \boxed{x^2 \le 1} \end{array}\right.$$

$$\Rightarrow \|(x,y)\| = \sqrt{x^2+y^2} \le \sqrt{2} \quad \forall (x,y)\in D$$

Do Ex 1.7.12

For each set discuss if D is open, closed, bounded.

i) $D = \{(x,y) \in \mathbb{R}^2 : \underline{xy \overset{\downarrow}{>} 0}\}$

Sol: $D = \{f(x,y) > 0\}$ where $f(x,y) = xy \in \mathcal{C}(\mathbb{R}^2)$

$\Rightarrow$ D is open.

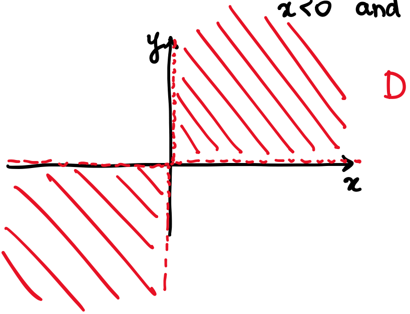
Is D bounded ?

Sol 1 : with the help of a figure

$xy > 0 \Leftrightarrow x > 0$ and $y > 0$

or

$x < 0$ and $y < 0$



D seems to be unbounded

From figure this is eviden

Sol 2: (without figure)

Rmk (gen. rmk on unbounded sets)

How can I prove that D is unbdd ?

D bdd $\Leftrightarrow \exists M : \underline{\|\vec{x}\| \leq M} \ \forall \vec{x} \in D$

D unbdd $\Leftrightarrow \nexists M : \|\vec{x}\| \leq M \ \forall \vec{x} \in D$

$$\forall M \ \exists \vec{x} \in D : \|\vec{x}\| > M \leftarrow$$

$$\Leftrightarrow \forall n = 1,2,3,\dots \ \exists \vec{x}_n \in D : \|\vec{x}_n\| > n$$

$$\Leftrightarrow \exists (\vec{x}_n) \in D : \|\vec{x}_n\| \longrightarrow +\infty$$

$$\boxed{D \text{ unbded} \Leftrightarrow \exists (\vec{x}_n) \subset D : \vec{x}_n \longrightarrow \infty_d} \quad ■$$

So, if $D = \{xy > 0\}$, $\vec{x}_n := (n,n) \in D \ \forall n \in \mathbb{N}, n \geq 1$

$$\vec{x}_n \Big\downarrow {\scriptstyle n \to \infty} \quad \infty_2$$

$\Rightarrow$ D unbded. ■

$\left(1 < \cancel{xy < 2, \ x \leq y \leq 2x}\right)$

1.7.12

iv) $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq xy \leq 2, \ x \leq y \leq 2x\}$

Sol: D is defd through large ineq involving cont functs $\Rightarrow$ D is closed.

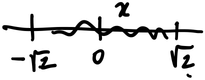
Bounded? Let's see if we can say anything without drawing D.

$$(x,y) \in D \Leftrightarrow \begin{cases} \boxed{1 \leq \boxed{xy \leq 2}} \leftarrow \\ \quad\ \downarrow \\ \therefore x \leq y \leq 2x \leftarrow \end{cases}$$

$\boxed{x>0} \quad \Rightarrow \quad \boxed{x^2 \le xy \le 2x^2}$

$\Downarrow$

$\underline{x^2 \le xy \le 2} \quad \Rightarrow \quad \boxed{x^2 \le 2}$



$\Rightarrow \quad x \le y \le 2x$

$-2\sqrt{2} \le \boxed{-\sqrt{2} \le y \le 2\sqrt{2}}$

$\Downarrow$

$\boxed{|y| \le 2\sqrt{2}} \quad \Rightarrow \quad \boxed{y^2 \le 8}$ ↙

$\boxed{x>0} \quad \Rightarrow \quad \boxed{\|(x,y)\| = \sqrt{x^2+y^2} \le \sqrt{10}} \leftarrow =: M$

$\boxed{x=0}$ no pts on D. $\left\{ 1 \le 0 \le 2 \right.$ imposs.

$\boxed{x<0}$ $D: \begin{cases} 1 \le xy \le 2 \\ x \le y \le 2x \end{cases}$ no $\underline{(x,y) \in D}$

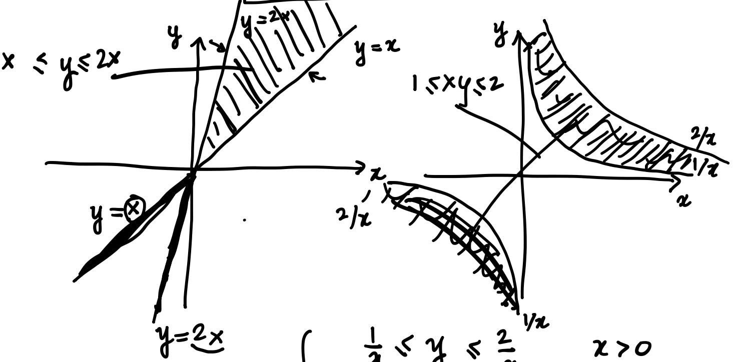
$\Uparrow$

$x \le y \le 2x$ no y

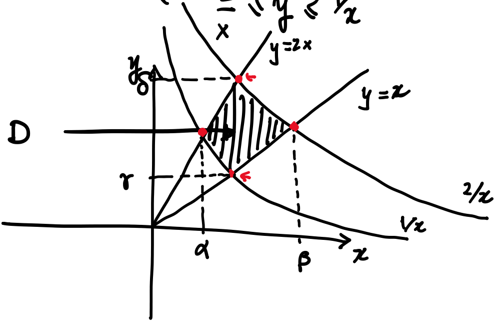
$\underset{\substack{\vee \\ 2x}}{x}$ bigger than

$\Rightarrow$ D bded.

Figure $\begin{cases} \boxed{1 \le xy \le 2} \\ \boxed{x \le y \le 2x} \Leftarrow \end{cases}$ $\Leftrightarrow \quad \underline{\frac{1}{x} \le y \le \frac{2}{x}}$ No!



$$1 \leq xy \leq 2 \iff \begin{cases} \frac{1}{x} \leq y \leq \frac{2}{x} & x>0 \\ 1 \leq 0 \leq 2 \text{ imposs} & x=0 \\ \frac{2}{x} \leq y \leq 1/x & x<0 \end{cases}$$



$$\begin{aligned} 0 < \alpha \leq x \leq \beta \quad &\Rightarrow \quad x^2 \leq \beta^2 \\ 0 \leq \gamma \leq y \leq \delta \quad &\Rightarrow \quad y^2 \leq \delta^2 \end{aligned} \Rightarrow \|(x,y)\| \leq \sqrt{\beta^2+\delta^2} =: M$$

$\Rightarrow$ D is bdd. ■

Do Ex 1.7.12 (compact means closed and bdd)