Ex 1.7.12

ii) $D = \{ (x,y) \in \mathbb{R}^2 : |xy| \leq 1 \}$

D open? closed? bounded? compact?

Def: We say that D is compact iff D is closed and bounded.

Sol: D is defd through a large ineq

$$\underbrace{1 - |xy|}_{f(x,y)} \geq 0$$

$f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow$ D is closed.

Can D be open?

General fact:

Prop: The unique sets of $\mathbb{R}^d$ that are both open and closed are $\phi, \mathbb{R}^d$.

In our case clearly $\boxed{D \neq \phi}$

(for ex. $(0,0) \in D$ because $|0 \cdot 0| \leq 1$ true)

and $\boxed{D \neq \mathbb{R}^d}$ (because if $|xy| > 1$

$\Rightarrow (x,y) \notin D$, for ex $(2,2)$ $|2 \cdot 2| > 1$ true)

$\Rightarrow$ D cannot be open.

Is D bounded?

Let's try with a figure:

$(x,y)\in D \iff |xy|\leqslant 1 \iff$

$$\boxed{-1\leqslant xy\leqslant 1}$$

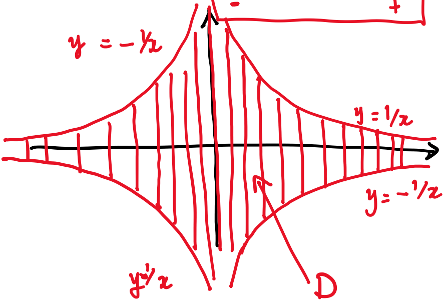
$$\iff \begin{cases} x>0 & \boxed{-\frac{1}{x}\leqslant y\leqslant 1/x} \\ x=0 & -1\leqslant 0\leqslant 1 \text{ true } \forall y : \boxed{\begin{array}{c}(0,y)\in D\\ \forall y\in\mathbb{R}\end{array}} \\ x<0 & \boxed{\underset{-}{\frac{1}{x}}\leqslant y\leqslant \underset{+}{\overset{\uparrow}{-}\frac{1}{x}}}. \end{cases}$$

$\Downarrow$

D is unbdd

$\vec{x}_n=(0,n)\to\underset{2}{\infty}$

$n\in\mathbb{N}$



$\Rightarrow$ D is closed but unbdd $\Rightarrow$

D is not compact. ∎

Do iii)

v) $D=\{(x,y,z)\in\mathbb{R}^3 : \boxed{x^2+2y^2+3z^4<4}\}$

D open? closed? _bded? compact?

Sol: D is defd by a strict inea

$$\underbrace{x^2+2y^2+3z^4-4}_{f(x,y,z)\in\mathscr{C}(\mathbb{R}^3)} \boxed{<} 0$$

$\Rightarrow$ D is open.

D closed: notice that $D \neq \emptyset$

$(0,0,0)\in D$ and $D \subsetneq \mathbb{R}^d$

$(1,1,1)\notin D$ because $\underbrace{1^2+2\cdot 1^2+3\cdot 1^4}_{\underbrace{1+2+3}_{6\,\not<\,4}} \not< 4$

$$\Downarrow$$

$$D \neq \emptyset, \mathbb{R}^d$$

$\Rightarrow$ D is not closed.

D bded:

$$(x,y,z)\in D \Leftrightarrow \boxed{\underset{+}{x^2}+\underset{+}{2y^2}+\underset{+}{3z^4}<4}$$

$$\Rightarrow \begin{array}{l} x^2<4 \\ \cancel{2}y^2<4/2 \\ \cancel{3}z^4<4/3 \Rightarrow z^2 \le \sqrt{\frac{4}{3}} \end{array}$$

$$\Rightarrow \|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$$

$$\le \sqrt{4+2+\sqrt{\tfrac{4}{3}}} =: M$$

$$\forall (x,y,z) \in D$$

$$\Downarrow$$

D is bounded

$\Rightarrow$ since D is not closed $\Rightarrow$
D cannot be compact ■

Do vi)

vii) $D = \{ (x,y,z) \in \mathbb{R}^3 : x^2 - y^2 + z^2 \circled{\leq} 1 \}$

Same questions.

Sol: D is def'd through a large ineq

$$\underbrace{x^2 - y^2 + z^2 - 1}_{f(x,y,z) \in \mathcal{C}(\mathbb{R}^3)} \leq 0$$

$\Rightarrow$ D is closed.

Is D open? Since $D \neq \phi$ $((0,0,0) \in D)$
and $D \neq \mathbb{R}^d$ $(100, 1, 0) \notin D$

$\Rightarrow$ D is not open.

Is D bded?

Here $(x,y,z) \in D \Leftrightarrow$

$$\overset{+}{x^2} \ominus \overset{-}{y^2} + \overset{+}{z^2} \leq 1$$

$$\cancel{\begin{cases} x^2 \le 1 \\ -y^2 \le 1 \\ z^2 \le 1 \end{cases}}$$

Guess: D is unbdd

I need to show that

$\exists (\vec{x}_n) \subset D$, $\vec{x}_n \longrightarrow \infty_3$

$(n, n, 0) \in D$ yes because

- $\underbrace{n^2 - n^2 + 0^2}_{=0} \le 1$ true

- $(n,n,0) \xrightarrow{n \to +\infty} \infty_3$

$$\left(\|(n,n,0)\| = \sqrt{n^2+n^2+0^2} = \sqrt{2}\, n \longrightarrow +\infty \right)$$

$\Rightarrow$ D is unbdd

$\Rightarrow$ D is not compact. ■

Do viii).

Weierstrass Thm

$\boxed{f \in \mathcal{C}(D)}$

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, D closed and bounded (that is, D compact). Then f has both global min pt and global max pt on D, that is

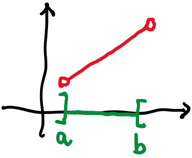
$$\exists \underbrace{\vec{x}_{min}}_{\in \mathbb{R}^d}, \vec{x}_{max} \in \underbrace{D} \ : \ \underbrace{f(\vec{x}_{min})}_{\in \mathbb{R}} \leq f(\vec{x}) \leq f(\vec{x}_{max}) \quad \forall \vec{x} \in D$$

Value $f(\vec{x}_{min})$ is called min value of f on D

" $f(\vec{x}_{max})$ " max " " "

What about the case of non compact domain? In general, we cannot expect to have any min/max. This is well known in the case $d = 1$.

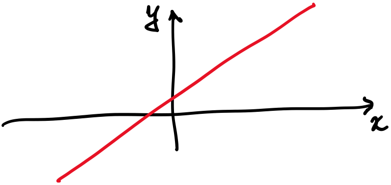
Ex: $f = f(x)$ $x \in D \subset \mathbb{R}$.



$$f \doteq f(x) : \overset{D}{]a,b[} \longrightarrow \mathbb{R}$$

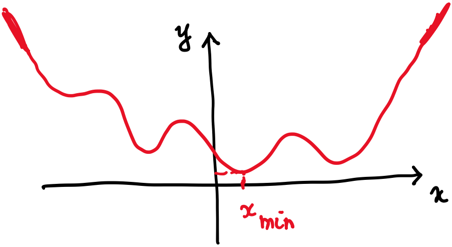
$f \in \mathcal{C}(]a,b[)$ here f has not $x_{min}, x_{max} \in]a,b[$

$f = f(x)$ $x \in D \subset \mathbb{R}$ $D = \mathbb{R}$



So, in gen, nothing can be said about $\min_{D} f$, $\max_{D} f$ if D is not compakt. However, there is an important case in which some thing general can be said.

Ex: $f = f(x) : D = \mathbb{R} \longrightarrow \mathbb{R}$, $f \in \mathscr{C}(D)$



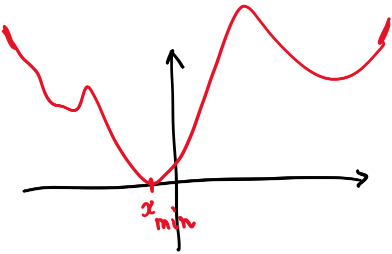
$$\lim_{x \to \pm\infty} f(x) = +\infty$$

$$\Downarrow$$

$\exists\ x_{min} \in \mathbb{R}$:

$f(x_{min}) \leq f(x)$ $\forall\ x \in \mathbb{R}$

$\nexists\ x_{max}$.



Thm (Corollary of W. thm)

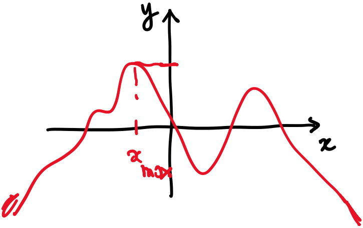
Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $f \in \mathscr{C}(D)$, D closed and unbounded. Assume also that (Ex: $D = \mathbb{R}^d$)

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty$$

Then f has global min pt (but not global max pt ✓

$$\exists \vec{x}_{min} \in D : \quad f(\vec{x}_{min}) \le f(\vec{x}) \quad \forall \vec{x} \in D$$

Dual statement



Prop: Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $f \in \mathcal{C}(D)$, D closed and unbounded. Assume that

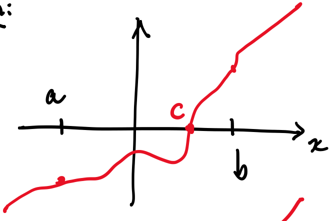
$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = -\infty .$$

Then there exists global max point for f on D (but no global min pts), that is

$$\exists \vec{x}_{max} \in D : \quad f(\vec{x}_{max}) \ge f(\vec{x}), \quad \forall \vec{x} \in D.$$

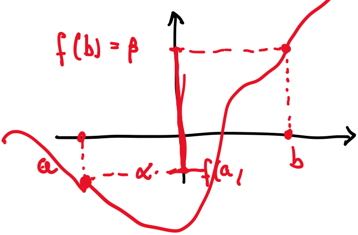
Intermediate Values Thm

Refresh:



Zeroes Thm

$f \in \mathcal{C}(I)$, I interv



Int. values Thm

$f\in\mathcal{C}(I)$, I interval

$\alpha,\beta\in f(I) \Rightarrow$

$\underline{[\alpha,\beta]}\subset f(I)$

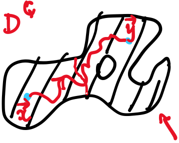
Thm: $\boxed{f\in\mathcal{C}(I)}$ $\underline{I}\subset\mathbb{R}$ interval. Then

$\underline{f(I)}$ is an interval

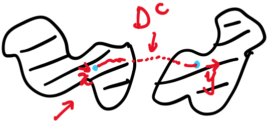
What happens if $f=f(\vec{x})$.

Def: (connected set)

Idea: a connected set is a subset of $\mathbb{R}^d$ made of one single "piece"



connected



not connected

Precisely: Let $D\subset\mathbb{R}^d$, We say that D is connected if

$\vec{\gamma}\in\mathcal{C}(I)$

$\vec{\gamma}=\vec{\gamma}(t): \overset{[a,b]}{I}\subset\mathbb{R}\longrightarrow\mathbb{R}^d$

$\forall\, \vec{x},\vec{y}\in D$, $\exists$ continuous path joining $\vec{x}$ to $\vec{y}$ entirely contained in D

$\vec{\gamma}(a)=\vec{x}$

$\vec{\gamma}(b)=\vec{y}$

$\vec{\gamma}(t)\in D$ $\forall t\in[a,b]$

that is

$\forall \vec{x}, \vec{y} \in D \quad \exists\, \vec{\gamma} = \vec{\gamma}(t) : [a,b] \to \mathbb{R}^d$:

i) $\vec{\gamma} \in \mathcal{C}([a,b])$

ii) $\vec{\gamma}(a) = \vec{x}$, $\vec{\gamma}(b) = \vec{y}$

iii) $\vec{\gamma}(t) \in D$, $\forall t \in [a,b]$

Thm: Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}$, D connected set, $f \in \mathcal{C}(D)$.

Then,

$f(D)$ is an interval.

Differential Calculus

To solve optimization pbs we need to introduce a concept of derivative for

$$f = f(\vec{x}).$$

More in general, we now discuss the pb of defining a derivative for

$$\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}^m$$

If

$$f = f(x) : D \subset \mathbb{R} \to \mathbb{R}$$

then

$$\mathbb{R} \ni f'(x) := \lim_{h\to 0} \boxed{\frac{f(x+h)-f(x)}{h}} \in \mathbb{R}$$

What if we try to def this but with $f(x)$ replaced by $\vec{F}(\vec{x}) \in \boxed{\mathbb{R}^d}$

$$? \ni \vec{F}'(\vec{x}) := \lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h})-\vec{F}(\vec{x})}{\boxed{\vec{h}}}$$

But: what is $\boxed{\in\mathbb{R}^m}$ $\boxed{\in\mathbb{R}^m}$

$$\frac{\boxed{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}}{\vec{h} \in \boxed{\mathbb{R}^d}} \boxed{\in \mathbb{R}^m} \,?$$

that is how can we compute

$$\frac{\text{vect of } \mathbb{R}^m}{\text{vect of } \mathbb{R}^d}$$

Pb is that there is not such operation!

Rmk: Nothing changes if $m = 1$

so if we consider $f = f(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}$

because

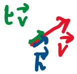
$$\lim \frac{f(\vec{x}+\vec{h})-f(\vec{x})}{\boxed{\vec{h}}} \quad \boxed{\frac{\in\mathbb{R}}{\mathbb{R}^d}} \,? \qquad \vec{h}\in\mathbb{R}^d$$

Directional Derivative

Dir. deriv. is a way to bypass the technical difficulty with a "natural trick"

$$\vec F'(\vec x) := \lim_{\vec h\to\vec 0} \frac{\vec F(\vec x+\vec h)-\vec F(\vec x)}{\vec h}$$

Idea: $\begin{pmatrix}\vec h\\ \downarrow\\ \vec 0\end{pmatrix} = \begin{pmatrix}\boxed{t\,\boxed{\vec v}}\\ \downarrow\\ 0\end{pmatrix}$ fixed



$$\text{"="} \lim_{\boxed{t\to 0}} \frac{\vec F(\vec x+t\vec v)-\vec F(\vec x)}{t\cancel{\vec v}} \quad \frac{\in\mathbb R^m}{\in\mathbb R}$$

Def: $\vec F = \vec F(\vec x) : D\subset\mathbb R^d \longrightarrow \mathbb R^m$

$$\mathbb R^m \ni D_{\vec v}\vec F(\vec x) := \lim_{t\to 0} \frac{\vec F(\vec x+t\vec v)-\vec F(\vec x)}{t} \quad \frac{\in\mathbb R^m}{\in\mathbb R}$$

directional derivative of $\vec F$ at pt. $\vec x$ along direction $\vec v$

Ex 2.1.2

Compute $D_{(1,1)} f(0,0)$ where $f(x,y) = x\cos y$

Sol: By def

$$D_{(1,1)} f\overset{\vec x}{(0,0)} := \lim_{t\to 0} \frac{f\big(\underbrace{(0,0)+t(1,1)}_{\vec x+t\vec v\,=\,(t,t)}\big) - f\overset{\vec x}{(0,0)}}{t}$$

$\vec{v}$

$$= \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{t} \cos t - \cancel{0}}{\cancel{t}}$$

$$= \lim_{t \to 0} \cos t = \boxed{1}$$

$$\Rightarrow \quad \exists \; D_{(1,1)} f(0,0) = 1. \quad \blacksquare$$

Do Ex 2.6.1

i) $D_{(\sqrt{3},1)} f(1,1)$. $f(x,y) = \log(1+xy) \leftarrow$

Sol: $$\underbrace{D_{(\sqrt{3},1)}}_{\boxed{\vec{v}}} \overset{\boxed{\vec{x}}}{f(1,1)} = \lim_{t \to 0} \frac{f\big(\overset{(1,1)+(t\sqrt{3},\,t)}{(1,1)+t(\sqrt{3},1)}\big) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{f(1+t\sqrt{3},\, 1+t) - f(1,1)}{t}$$

$$= \lim_{t \to 0} \frac{\log\big(1 + \overset{1+t(\sqrt{3}+1)+t^2\sqrt{3}}{(1+t\sqrt{3})(1+t)}\big) - \log 2}{t}$$

$$= \lim_{t \to 0} \frac{\overset{\log 2}{\overset{\uparrow}{\log}}\big(2 + t\,\overset{\nearrow 0}{(\sqrt{3}+1) + t^2\sqrt{3}}\big) \boxed{-\log 2}}{\boxed{t} \to 0} \to 0 \quad \left(\frac{0}{0}\right)$$

$$\boxed{\overset{H}{=}}\lim_{t\to 0}\frac{\frac{1}{2+t(\sqrt{3}+1)+t^2\sqrt{3}}(\sqrt{3}+1+2\sqrt{3}t)}{1}$$

$$=\lim_{t\to 0}\frac{\sqrt{3}+1+\boxed{2\sqrt{3}\,t}\nearrow 0}{2+\underbrace{t(\sqrt{3}+1)}_{\downarrow\, 0}+\underbrace{t^2\sqrt{3}}_{\downarrow\, 0}}\quad\begin{matrix}\to\sqrt{3}+1\\ \to 2\end{matrix}$$

$$=\frac{\sqrt{3}+1}{2}$$

$$\Rightarrow\quad\exists\quad D_{(\sqrt{3},1)}f(1,1)=\frac{\sqrt{3}+1}{2}.\ \blacksquare$$

Ex 2.6.1

iv) $D_{(-1,1)}f(0,0)$ where

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^4} & (x,y)\neq\vec{0}\ \leftarrow\\ 0 & (x,y)=\vec{0}\ \leftarrow\end{cases}$$

Sol: We have

$$(0,0)+(-t,t)=(-t,t)$$

$$\underline{D_{\underset{\vec{v}}{(-1,1)}}f(\overset{\vec{x}}{0,0})}=\lim_{t\to 0}\frac{f((0,0)+\overset{\shortparallel}{t(-1,1)})-f(0,0)}{t}$$

$$=\lim\quad\boxed{f(-t,t)}-\overset{0}{\overset{\shortparallel}{f(0,0)}}$$

$$\lim_{\substack{t\to \textcircled{0}\\ t\neq 0\\ (-t,t)\neq\vec{0}}} \qquad \frac{}{t}$$

$$= \lim_{t\to 0} \frac{\dfrac{-t\cdot\cancel{t}}{(-t)^2+t^4} - \cancel{0}}{\cancel{t}}$$

$$= \lim_{t\to 0} -\frac{t}{t^2+t^4} = \lim_{t\to 0} -\frac{1}{t+t^3}$$

$$\cdot \lim_{t\to 0+} \ominus \frac{1}{\underset{\substack{\downarrow\\ 0+}}{t}\,\underset{\substack{\downarrow\\ 1}}{\underbrace{(1+t^2)}}} \qquad \to 0+ \qquad = -\infty$$

$$\cdot \lim_{t\to 0-} \ominus \frac{1}{\underset{\substack{\downarrow\\ 0-}}{t}\,\underset{\substack{\downarrow\\ 1}}{(1+t^2)}} \nearrow -\infty \qquad \longrightarrow 0- \qquad = +\infty$$

$$\Rightarrow \nexists \lim_{t\to 0} -\frac{1}{t+t^3}$$

$$\Rightarrow \nexists\ D_{(-1,1)} f(0,0). \qquad \blacksquare$$

Do all others of 2.6.1.