Monday, 27 February 2023 08:29

$\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^{\textcircled{d}} \longrightarrow \mathbb{R}^{\textcircled{m}}$ is differentiable at $\vec{x}$ if $\exists\ \vec{F}'(\vec{x})$ $m \times d$ matrix (jacobian matrix) s.t.

$$\boxed{\boxed{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h}} = \underbrace{o(\vec{h})}} \qquad (*)$$

$$\Leftrightarrow \boxed{\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h}}{\|\vec{h}\|} = \vec{0}}$$

$$\vec{F}'(\vec{x}) = \begin{bmatrix} \nabla f_1(\vec{x}) \\ \nabla f_2(\vec{x}) \\ \vdots \\ \nabla f_m(\vec{x}) \end{bmatrix}$$

$\vec{F} = (f_1, \dots, f_m) \qquad f_j = f_j(\vec{x})$

$$\nabla f_j = (\partial_1 f_j, \partial_2 f_j, \dots, \partial_d f_j)$$

If $m = 1$, $\cancel{\vec{F}} = \cancel{\vec{F}}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^1 = \mathbb{R}$

$$F'(\vec{x}) = \nabla F(\vec{x}) = (\partial_1 F(\vec{x}), \dots, \partial_d F(\vec{x}))$$

Rmk: To write $\vec{F}'(\vec{x})$, we need the existence of all $\partial_i f_j$ $\quad \vec{F} = (f_1 — f_m)$

$\forall i, j$

$$\underline{\vec{F}'(\vec{x})} = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 \cdots & \partial_d f_1 \\ \partial_1 f_2 & \partial_2 f_2 \cdots & \partial_d f_2 \\ \vdots & & \\ \partial_1 f_m & \partial_2 f_m \cdots & \partial_d f_m \end{bmatrix}$$

We might think that $\exists\ \partial_i f_j \quad \forall\, i,j$

$$\Downarrow \quad ?$$

$$\exists\ \vec{F}'(\vec{x})$$

Unfortunately, the sole $\exists\ \partial_i f_j \ \forall\, i,j$ is NOT SUFFICIENT to imply DIFFERENTIABILITY

This is because

Prop:
If $\vec{F}$ is diffentiable at $\vec{x}$ $\Rightarrow$ $\vec{F}$ is cont at $\vec{x}$

Then, reminding of an ex where we have $\boxed{\exists\ D_{\vec{v}} f(0,0)\ \forall \vec{v}}$, but f is not cont at $(0,0)$

we have, in part, an f s.t.

- $\exists\ D_{(1,0)} f(0,0) = \partial_x f(0,0)$

  $\exists\ D_{(0,1)} f(0,0) = \partial_y f(0,0)$

$\Rightarrow \exists\ \nabla f = (\partial_x f, \partial_y f)$.

$\Rightarrow$ the jac. matrix can be written

- but f is not cont at $(0,0)$

$$\Downarrow$$

$$\rightarrow \boxed{f \text{ is not diff at } (0,0)}$$

$\blacksquare$

Ex 2.2.3 Discuss diff at $(0,0)$ for

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} \leftarrow & \boxed{(x,y)\neq\vec{0}} \\ \boxed{0} & (x,y)=\vec{0} \leftarrow \end{cases}$$

Sol: Here $f = f(x,y) : D = \mathbb{R}^2 \longrightarrow \mathbb{R}^1$

To check if f is $(d=2,\ m=1)$ diff at $(0,0)$ we start computing the jac matrix

$$f'(0,0) = [\partial_x f(0,0) \quad \partial_y f(0,0)] = \nabla f(0,0)$$

$$\partial_x f(0,0) \overset{?}{=}$$

$$\partial_x f(x,y) \underset{\boxed{(x,y)\neq\vec{0}}}{=} \partial_x\left(\frac{\boxed{x^2}\boxed{y^2}}{x^2+y^2}\right)$$

$$= y^2\,\frac{2x(x^2+y^2) - x^2\,2x}{(x^2+y^2)^2}$$

however this is useless to compute $\partial_x f(0,0)$. (I would need to take $x=y=0$ but I can't because $(x,y)\neq\vec{0}$)

$$\partial_x f(0,0) = D_{\underline{(1,0)}} f(0,0)$$

$$= \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t}$$

$$= \lim_{\substack{t\to 0\\ t\neq 0}} \frac{\overset{0}{\overset{\|}{f(t,0)}} - \overset{0}{\overset{\|}{f(0,0)}}}{t} \;=0\; = \lim_{t\to 0}\frac{0}{t}$$

$$= 0$$

$$\Rightarrow \quad \exists \;\; \partial_x f(0,0) = 0$$

Some calc. shows $\partial_y f(0,0) = 0$

$$\Rightarrow \quad \nabla f = (0,0) = f'(0,0)$$

Is this enough to conclude f is diff at $(0,0)$? No! By what we have seen in prev. rmk, the $\exists\ \nabla f \not\Rightarrow f$ diff.

So, how can we check if $\underline{f \text{ is diff}}$?

By the def: that is

$$\underset{\substack{\|\\(0,0)}}{f(\vec{x}} + \underset{\substack{\|\\(u,v)}}{\vec{h})} - \boxed{\underset{\substack{\|\\(0,0)}}{f(\vec{x})}} - \underline{[0 \quad 0]}\,\vec{h} \overset{?}{=} o(\vec{h}) \leftarrow$$

$$\Leftrightarrow \quad f((0,0)+(u,v)) - f(0,0) - [0 \quad 0]\begin{pmatrix}u\\v\end{pmatrix} = o\left(\begin{pmatrix}u\\v\end{pmatrix}\right)$$

$$\Leftrightarrow \quad f(u,v) - f(0,0) - \overrightarrow{[0 \quad 0]}\boxed{\begin{pmatrix}u\\v\end{pmatrix}} = \underline{o\begin{pmatrix}u\\v\end{pmatrix}}$$

$$\Leftrightarrow \quad \frac{f(u,v) - f(0,0) - \underline{[0\ \ 0]\begin{pmatrix}u\\ v\end{pmatrix}}}{\|(u,v)\|} \xrightarrow[\vec{h}=(u,v)\to\vec{0}]{?} 0$$

$$\Leftrightarrow \quad \underset{\substack{(u,v)\to(0,0)\\ \Updownarrow}}{\lim} \frac{\boxed{f(u,v)} - \overset{\overset{0}{\shortparallel}}{f(0,0)} - 0}{\sqrt{u^2+v^2}} = 0\ ?$$

$$\Leftrightarrow \quad \lim_{(u,v)\to(0,0)} \frac{\frac{u^2v^2}{\boxed{u^2+v^2}}}{\sqrt{u^2+v^2}} = 0\ ?$$

$$\Leftrightarrow \quad \lim_{(u,v)\to(0,0)} \frac{u^2v^2}{(u^2+v^2)^{\underset{\underset{3/2}{\shortparallel}}{1+1/2}}} = 0\ ?$$

$$\underset{\substack{(u,v)\to(0,0)\\ \Updownarrow\\ \rho\to 0\\ 0\leq\theta\leq 2\pi}}{\lim} \quad \underset{g(u,v)}{\boxed{\frac{u^2v^2}{(u^2+v^2)^{3/2}}}}$$

$$g(u,v) \underset{\substack{u=\rho\cos\theta\\ v=\rho\sin\theta}}{=} \frac{\rho^{4}\cos^2\theta\ \cancel{\rho^2}\sin^2\theta}{(\rho^2)^{3/2}}$$

$$= \frac{\rho^4}{\rho^3}\cos^2\theta\ \sin^2\theta$$

$$= \underset{\substack{\downarrow\\ 0}}{\boxed{\rho}} \cdot \underset{0\leq \qquad\qquad \leq 1}{\boxed{\cos^2\theta\ \ \sin^2\theta}}$$

$$\begin{array}{ccccc} 0 & \le & \boxed{|g(u,v)|} & \le & \rho \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \qquad (u,v) \to (0,0)$$

Cond: $\exists \lim\limits_{(u,v)\to\vec{0}} g = 0 \Rightarrow f$ is diff at $(0,0)$. ∎

Do Ex 2.6.2

Thm (diffentiability test)

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$,

$$\vec{F}(\vec{x}) = \left(f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x})\right)$$

If

$$\boxed{\partial_{\underline{i}} f_j \in \mathcal{C}(D)} \quad \begin{array}{l} \forall \; i = 1 \dots d \\ \forall \; j = 1 \dots n \end{array}$$

$\Rightarrow \vec{F}$ is diff on D.

Ex: $f(x,y) = x^2 + xy : D = \mathbb{R}^2 \longrightarrow \mathbb{R}$

Here

$$\partial_x f(x,y) = 2x + y \in \mathcal{C}(\mathbb{R}^2)$$

$$\partial_y f(x,y) = x \in \mathcal{C}(\mathbb{R}^2)$$

(diff. test)

$\Rightarrow f$ is diff on $\mathbb{R}^2$

$$\Downarrow$$

$$f'(x,y) = \nabla f(x,y) = (2x + y, \; x)$$ ∎

Some rules of Calc.

$$\vec{F}, \vec{G} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m \quad \exists\, \vec{F}', \vec{G}'$$

$$(\vec{F} \pm \vec{G})' = \vec{F}' \pm \vec{G}'$$

$$f, g : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$\nabla (fg) = g \nabla f + f \nabla g$$

Chain Rule:

$$\begin{tikzcd} \vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^{\textcircled{d}} \arrow[r] \arrow[dr] & \mathbb{R}^{\textcircled{m}} \\ \vec{G} = \vec{G}(\vec{y}) : \tilde{D} \subset \mathbb{R}^{\textcircled{m}} \arrow[r] & \mathbb{R}^{\textcircled{k}} \end{tikzcd}$$

$$\vec{G} \circ \vec{F} = \vec{G}(\vec{F}(\vec{x})) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^k$$

$$\left(\vec{G}(\vec{F}(\vec{x})) \right)' = \overbrace{\underbrace{\vec{G}'(\vec{F}(\vec{x}))}_{k \times \textcircled{m}} \underbrace{\vec{F}'(\vec{x})}_{\textcircled{m} \times d}}^{k \times d} \leftarrow$$

$$\left(\cancel{\vec{F}'(\vec{x})\, \vec{G}'(\vec{F}(\vec{x}))} \right)$$

Special case: we have

$$g = g(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^1$$

$$\vec{\gamma} = \vec{\gamma}(t) : I \subset \mathbb{R}^1 \longrightarrow \mathbb{R}^d$$

$$\Rightarrow \phi(t) := g(\vec{\gamma}(t)) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$$

$$\boxed{\frac{d}{dt}\phi(t)} = \underbrace{g'(\vec{\gamma}(t))}_{\underbrace{\left[\nabla g(\vec{\gamma}(t))\right]}} \underbrace{\vec{\gamma}\,'(t)}_{\begin{pmatrix}\gamma_1'(t)\\ \gamma_2'(t)\\ \vdots \\ \gamma_d'(t)\end{pmatrix}}$$

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_{\underset{\smile}{d}}(t))$$

$$= \nabla g(\vec{\gamma}(t)) \underset{\substack{\uparrow \\ \text{scalar prod}}}{\cdot} \vec{\gamma}\,'(t)$$

$$\boxed{\frac{d}{dt}\enclose{circle}{g}(\vec{\gamma}(t)) = \nabla g(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)}$$

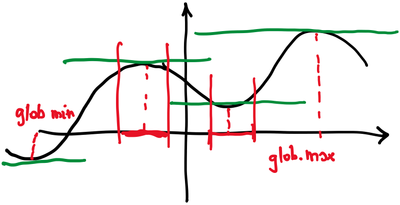
(total derivative)

· Optimization Problems

A very important pb in applications is the search of min / max for $f = f(\vec{x})$ $\vec{x} \in D$:

$$\min_{D} f(\vec{x}) \quad , \quad \max_{D} f(\vec{x}) .$$

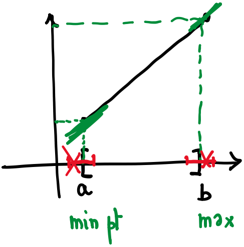
Case $\quad f = f(x) \quad x \in D \subset \mathbb{R}$



Roughly: at min/max pts

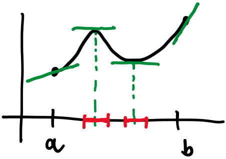
$$\boxed{f'(x)=0}$$

Roughly means that for a precise statement we must be more careful:



but $f'(x) \neq 0$

$\forall x \in [a,b]$



$\boxed{f'(x)=0}$ at min/max holds when min/max pts belong to Int D

Def: (local / global $\boxed{\text{min}}$/max pt)

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}$. A pt $\vec{x}_{min} \in D$ is called

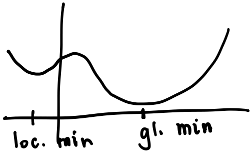
- global min. pt for f on D if

$$f(\vec{x}_{min}) \leq f(\vec{x}), \quad \forall \vec{x} \in D$$

- local min. pt. for f on D if

$$\exists B(\vec{x}_{min}, r] : \quad f(\vec{x}_{min}) \leq f(\vec{x}) \quad \forall \vec{x} \in D \cap B(\vec{x}_{min}, r]$$

Rmk: Every global min is also a local min but not vice versa



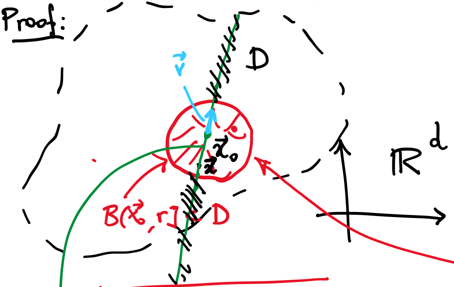
Rmk: Analogous def for glob/loc. max pts.

Thm (Fermat)

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}^1$, f differentiable on D. Then, if $\vec{x}_0$ is a local min (or max) pt for f on D and $\boxed{\vec{x}_0 \in \text{Int } D}$,

$$[\nabla f] = f'(\vec{x}_0) = \underset{\substack{\uparrow \\ \text{matrix} \\ 1 \times d}}{0} \iff \boxed{\nabla f(\vec{x}_0) = \vec{0}}$$

Proof:



We assume $\vec{x}_0$ is a loc. min for f on D

$\vec{x}_0 \in \text{Int } D$

$$\boxed{f(\vec{x}_0) \le f(\vec{x})} \quad \forall \vec{x} \in \underbrace{D \cap B(\vec{x}_0, r]}_{\subset D} = B(\vec{x}_0, r]$$

$$\vec{\gamma}(t) = \boxed{\vec{x}_0 + t\vec{v}} \qquad t \in \mathbb{R}$$

$$\boxed{\vec{\gamma}(0) = \vec{x}_0}$$

t small, $\tilde{\gamma}(t) \approx \vec{x}_0$ $\Rightarrow$ $\vec{\gamma}(t) \in B(\vec{x}_0, r]$

($\vec{\gamma}(t) \in B(\vec{x}_0, r] \Leftrightarrow \|\vec{\gamma}(t) - \vec{x}_0\| \le r$

$$\Leftrightarrow \|\cancel{\vec{x}_0} + t\vec{v} - \cancel{\vec{x}_0}\| \le r$$

$$\Leftrightarrow |t|\,\|\vec{v}\| \le r$$

$$\Leftrightarrow \underline{|t| \le \frac{r}{\|\vec{v}\|}}.$$)

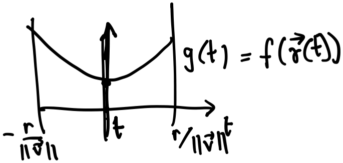
$\Rightarrow \underline{f(\vec{\gamma}(0))} = \underline{f(\vec{x}_0)} \le f(\vec{\gamma}(t))$ $\quad \forall t : |t| \le \frac{r}{\|\vec{v}\|}$

$\Rightarrow$ $f(\gamma(0)) \le f(\vec{\gamma}(t))$ " "

Def: $\underset{\in\mathbb{R}}{g(t)} := f(\vec{\gamma}(t)) \in \mathbb{R}$

$g(0) \le \underline{g(t)}$ $\quad \forall t : \underline{|t| \le \frac{r}{\|\vec{v}\|}}$

$\Rightarrow$ t = 0 is min for g on $t \in \underbrace{\left[-\frac{r}{\|\vec{v}\|}, \frac{r}{\|\vec{v}\|}\right]}$

→ 

$\Rightarrow$ $\boxed{g'(0) = 0}$ (Fermat, one dim vers)

$$g'(t) = \frac{d}{dt} g(t) = \underbrace{\frac{d}{dt} f(\vec{\gamma}(t))}_{\text{tot deriv}}$$

$$= \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)$$

$$\vec{\gamma}(t) = \vec{x}_0 + t\vec{v} \quad \Rightarrow \quad \vec{\gamma}\,'(t) = \vec{v}$$

$$\Rightarrow \quad g'(t) = \nabla f(\vec{\gamma}(t)) \cdot \vec{v}$$

$$\downarrow$$

$$\Rightarrow \quad \boxed{0 = g'(0)} = \nabla f(\vec{\gamma}(0)) \cdot \vec{v}$$

$$\parallel$$

$$= \nabla f(\vec{x}_0) \cdot \vec{v}$$

$$\Rightarrow \quad \nabla f(\vec{x}_0) \cdot \vec{v} = 0 . \quad \forall\, \vec{v} \in \mathbb{R}^d .$$

Take $\vec{v} = \nabla f(\vec{x}_0)$

$$\Rightarrow \quad \nabla f(\vec{x}_0) \cdot \nabla f(\vec{x}_0) = 0$$

$$\Updownarrow$$

$$\nabla f(\vec{x}_0) \perp \nabla f(\vec{x}_0)$$

$$\Updownarrow$$

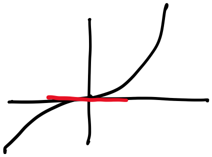
$$\nabla f(\vec{x}_0) = \vec{0} \qquad \blacksquare$$

Rmk 1: Condition $\nabla f(\vec{x}_0) = \vec{0}$ does not distinguish between a min and a max.

Rmk 2: There might be pts $\vec{x}_0$ where $\nabla f = \vec{0}$ but $\vec{x}_0$ is not a min nor a max.

Ex $d = 1$ $\quad f(x) = x^3 \quad f'(x) = 3x^2$

$f'(0) = 0$



but $x = 0$ is not min nor max

Def: We call stationary point any pt. $\vec{x}_0 \in D$ where $\nabla f(\vec{x}_0) = \vec{0}$.