Thursday, 2 March 2023 12:10

**Fermat Thm** $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$

diff on $D$, $\underline{\vec{x}_0 \in \text{Int } D}$ s.t.

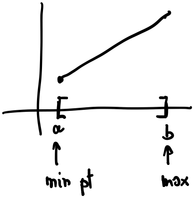
$\vec{x}_0$ is loc min/max

$$\Rightarrow \boxed{\nabla f(\vec{x}_0) = \vec{0}}$$

**Remarks:**

1. $\underline{\vec{x}_0 \in \text{Int } D}$ : if $\vec{x}_0 \notin \text{Int } D$ then
even if $\vec{x}_0$ is a loc min/max
$\nabla f(\vec{x}_0)$ can be $\neq \vec{0}$

$d = 1$



$f' \neq 0$ always

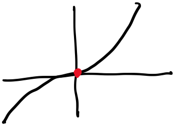
$\Downarrow$

$f'(a) \neq 0$

$f'(b) \neq 0$

2. $\nabla f = \vec{0}$ does not distinguish between min/max pts and it could be verified by pts which are not neither mins nor maxs.

$d = 1$



$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$

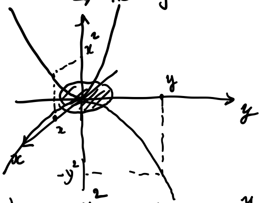
but $x = 0$ is not min/max

$d = 2$ $f(x,y) = x^2 - y^2$, $D = \mathbb{R}^2$

f is diff on D $\left( \begin{array}{l} \partial_x f = 2x \in \mathcal{C} \\ \partial_y f = -2y \in \mathcal{C} \\ \Downarrow \text{ diff test} \\ f \text{ is diff on } \mathbb{R}^2 \end{array} \right)$

$$\boxed{f(x,0) = x^2} \xrightarrow[x \to \pm\infty]{} +\infty$$

$\Rightarrow$ no global max is poss.



$$f(0,y) = -y^2 \longrightarrow -\infty \quad y \to \pm\infty$$

$\Rightarrow$ no global min

$$\nabla f = (2x, -2y) = \vec{0} \iff \begin{cases} 2x = 0 \\ -2y = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\Rightarrow \nabla f = \vec{0} \iff (x,y) = (0,0)$$

Is $(0,0)$ a $\overset{\text{loc.}}{\text{min}}$ ?

To be a loc. min we need

$$f(x,y) \geq \underset{\overset{\shortparallel}{0}}{f(0,0)} \quad \forall (x,y) \in B((0,0),r]$$

$$\Rightarrow f(x,y) \geq 0 \quad \forall (x,y) \in B((0,0),r]$$

but this is imposs because

$0 > -y^2 = f(0,y) \geq 0$ imposs !

Similarly, $(0,0)$ is not a loc. max, otherwise

$$f(x,y) \leq f(0,0) = 0 \quad \forall (x,y) \in B(\vec{0}, r]$$

but $0 < x^2 = f(x,0) \leq 0$ !

$x \neq 0$

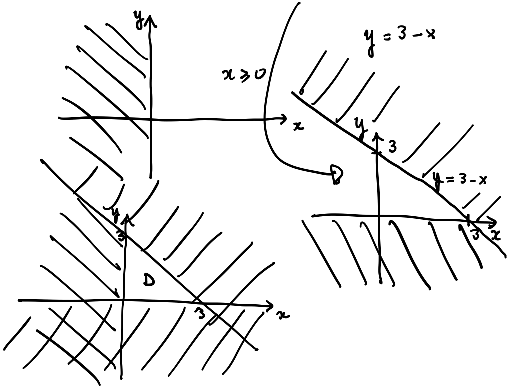
Point $(0,0)$ is called saddle point

Ex 2.3.5 Det min/max (if any) for $f(x,y) := x + y - xy$ on

$D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 3 - x,\ x \ge 0\}$

Sol:

$$D = \{(x,y) \in \mathbb{R}^2 : \boxed{0 \le y \le 3-x},\ \boxed{x \ge 0}\}$$



$\Rightarrow$ D is closed (because defd by large ineq.)

and bded $(0 \le x \le 3,\ 0 \le y \le 3)$

$+\ f \in \mathcal{C}(D) \Rightarrow$ (Weierstrass) $\exists$ both glob min/max for $f$ on $D$.

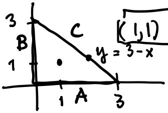
Now : let $(x,y) \in D$ be any global min/max pt. We have

- either $(x,y) \in \text{Int } D \Rightarrow$ (Fermat)

$$\nabla f = \vec{0}$$

$$\nabla f = (1-y,\ 1-x) = \vec{0}$$

$$\Leftrightarrow \begin{cases} 1-y=0 \\ 1-x=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=1 \end{cases}$$



$\boxed{(1,1) \in \text{Int } D}$

- or $(x,y) \in D \setminus \text{Int } D = \partial D$ (boundary of D)

$\Downarrow$

$(x,y) \in A \cup B \cup C$

$A = \{ \underbrace{(x,0)} : \underline{0 \leq x \leq 3} \}$

$B = \{ (0,y) : 0 \leq y \leq 3 \}$

$C = \{ (x, 3-x) : \underline{0 \leq x \leq 3} \}$

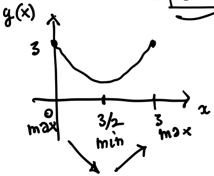
On A: $f(x,0) = x$ min on A $x=0 \Rightarrow \boxed{(0,0)}$
max on A $x=3 \Rightarrow \boxed{(3,0)}$

On B: $f(0,y) = y$ min on B $y=0 \Rightarrow \boxed{(0,0)}$
max on B $y=3 \Rightarrow \boxed{(0,3)}$

On C: $f(x,3-x) = \cancel{x} + 3 - \cancel{x} - x(3-x)$

$$= 3 - x(3-x)$$

$$= \underbrace{\boxed{3 - 3x + x^2}}_{g(x)} \qquad x \in [0,3]$$



$\boxed{\begin{matrix} g(0) = 3 \\ g(3) = 3 \end{matrix}}$

$g' = -3 + 2x > 0$

$\Leftrightarrow \quad x > 3/2$

ⓕ is min when $x = 3/2 \Rightarrow \left(\frac{3}{2}, 3 - \frac{3}{2}\right) = \boxed{\left(\frac{3}{2}, \frac{3}{2}\right)}$

" max " $x = 0, 3 \Rightarrow \boxed{(0,3), (3,0)}$

So: if $(x,y)$ is min for $f$ on $D$

$$\Rightarrow \quad (x,y)\in \operatorname{Int} D \Rightarrow (x,y)=(1,1) \xrightarrow{f} \boxed{1}$$

$$(x,y)\notin \operatorname{Int} D \Rightarrow \begin{array}{l} (x,y)\in A \Rightarrow (x,y)=\boxed{(0,0)} \to \boxed{0} \\ (x,y)\in B \Rightarrow \quad \text{″} \\ (x,y)\in C \Rightarrow (x,y)=\left(\frac{3}{2},\frac{3}{2}\right) \to 3-\frac{9}{4}=\boxed{\frac{3}{4}} \end{array}$$

$\Rightarrow$ $(0,0)$ is the global min.

If $(x,y)\in D$ is max for $f$ on $D$

$$\Rightarrow \quad (x,y)\in \operatorname{Int} D \Rightarrow (x,y)=(1,1) \xrightarrow{f} 1$$

$$\notin \operatorname{Int} D \Rightarrow \begin{array}{l} (x,y)\in A \Rightarrow (3,0) \xrightarrow{f} \boxed{3} \\ \in B \Rightarrow (0,3) \xrightarrow{f} \boxed{3} \\ \in C \Rightarrow (0,3),\ (3,0) \end{array}$$

$\Rightarrow$ $(0,3)$, $(3,0)$ are max pts for $f$ on $D$. ■

Ex. 2.6.4 Det. stationary pts of

i) $\boxed{f(x,y)=xy(x+1).} = x^2y+xy$

Sol: We have to find pts where

$$\nabla f(x,y)=\vec{0}$$

$$\nabla f = (\partial_x f, \partial_y f)$$

$$= (2xy+y,\ x^2+x) = \vec{0}$$

$(\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow$ (diff test) $f$ is diff on $\mathbb{R}^2)$

$$\nabla f = \vec{0} \Leftrightarrow \begin{cases} 2xy+y=0 \\ x^2+x=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \boxed{x(x+1)=0} \\ 2xy+y=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=-1 \\ -y=0 \end{cases}$$

$$\Leftrightarrow (0,0),\ (-1,0). \qquad \blacksquare$$

Do 2.6.4 #2, 3, (#4 is a bit tricky)

#5 $f(x,y,z) = \underline{(x^3-3x-y^2)}\, z^2 + z^3.$

Sol: Here $f$ is dfd on $\mathbb{R}^3$

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f)$$

$$= \left(z^2(3x^2-3),\ z^2(-2y),\ 2(x^3-3x-y^2)z+3z^2\right)$$

$$\left(\partial_x f, \partial_y f, \partial_z f \in \mathcal{C}(\mathbb{R}^3) \Rightarrow (\text{diff test})\ f \text{ is diff on } \mathbb{R}^3\right)$$

Now :

$$\boxed{\nabla f = \vec{0}} \Leftrightarrow \begin{cases} \cancel{3}\boxed{z^2(x^2-1) = 0} \leftarrow \\ \cancel{-2}\boxed{yz^2 = 0} \\ 2(x^3-3x-y^2)z+3z^2 = 0 \\ \uparrow \qquad \uparrow \end{cases}$$

$$\Leftrightarrow \begin{cases} z=0 \\ \cancel{0=0} \\ \cancel{0=0} \end{cases} \vee \begin{cases} x=\pm 1 \\ yz^2=0 \\ \end{cases}$$

$$\Downarrow \qquad\qquad \Updownarrow$$

$$\boxed{\begin{matrix}(x,y,0) \\ \forall\ x,y\in\mathbb{R}\end{matrix}} \qquad \begin{cases} x=1 \\ \boxed{yz=0} \\ 2(-2-y^2)z+3z^2=0 \end{cases} \vee \begin{cases} x=-1 \\ yz=0 \\ 2(2-y^2)z+3z^2=0 \end{cases}$$

$$\swarrow \quad \searrow \qquad\qquad \swarrow \quad \searrow$$

$$\begin{cases} x=1 \\ y=0 \\ \underline{-4z+3z^2=0} \end{cases} \vee \begin{cases} x=1 \\ z=0 \\ \cancel{0=0} \end{cases} \qquad \begin{cases} x=-1 \\ y=0 \\ 4z+3z^2=0 \end{cases} \vee \begin{cases} x=-1 \\ z=0 \\ 0=0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\begin{cases} x=1 \\ y=0 \\ z\underbrace{(3z-4)}=0 \end{cases} \vee \cancel{(1,y,0)} \quad y\in\mathbb{R} \qquad \begin{cases} x=-1 \\ y=0 \\ z(4+3z)=0 \end{cases} \vee \begin{cases} x=-1 \\ z=0 \\ - \end{cases} \cancel{(-1,y,0)}$$

$$\swarrow \quad \searrow \qquad\qquad\qquad \swarrow \quad \searrow$$

$$\begin{cases} x=1 \\ y=0 \\ z=0 \end{cases} \quad \begin{cases} x=1 \\ y=0 \\ z=4/3 \end{cases} \qquad \begin{cases} x=-1 \\ y=0 \\ z=0 \end{cases} \quad \begin{cases} x=-1 \\ y=0 \\ z=-4/3 \end{cases}$$

$$\cancel{(1,0,0)} \quad \boxed{(1,0,4/3)} \qquad \cancel{(-1,0,0)} \quad \boxed{(-1,0,-4/3)}$$

**Conclusion:** st. pts are

$$(x,y,0) \quad \forall x,y\in\mathbb{R}$$

$$(1,0,4/3)\ ,\ (-1,0,-4/3).$$

**Ex 2.6.6**

$$f(x,y) := x^2(1-y)$$

$$D = \{(x,y)\in\mathbb{R}^2 : \boxed{x^2+|y|\leq 4}\}$$

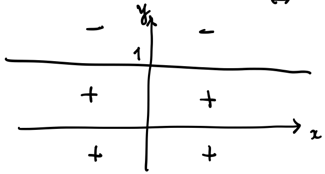
Q1. Det sign of $f$

→ Q2. Det min/max of $f$ on $D$ (if any)

Q3. Det $f(D)$

**Sol:** Q1 $\quad f(x,y)\geq 0 \Leftrightarrow x^2(1-y)\geq 0$

$$\Leftrightarrow 1-y\geq 0 \Leftrightarrow \boxed{y\leq 1}$$



Q2. D is clearly closed (because defd by a large ineq involving a cont funct) and bounded

$$\underbrace{x^2}_{+} + \underbrace{|y|}_{+} \le 4 \implies \begin{cases} |y| \le 4 \implies y^2 \le 16 \\ x^2 \le 4 \end{cases}$$

$$\Downarrow$$

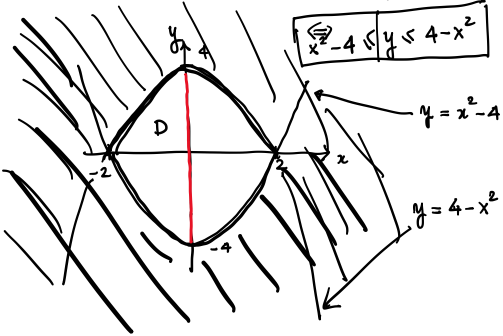
$$\|(x,y)\| = \sqrt{x^2+y^2} \le \sqrt{20}$$

$\forall (x,y) \in D$

Figure of D

$$x^2 + |y| \le 4 \iff |y| \le 4 - x^2$$

$$\boxed{\iff x^2 - 4 \le y \le 4 - x^2}$$



Since D is compact $\implies \exists$ both min & max pts for f on D.

**Determination**: If $(x,y) \in D$ is min/max for f

$\implies$ Case. 1: either $(x,y) \in \text{Int } D$

Case. 2: or $(x,y) \in D \setminus \text{Int } D = \partial D$

Case 1: 
$$\nabla f = (\partial_x f, \partial_y f)$$
$$= (2x(1-y), x^2(-1))$$
$$= (2x(1-y), -x^2)$$

Here we see that $\partial_x f, \partial_y f \in \mathcal{C}$

$\Rightarrow$ f is diff.

Hence, by Fermat thm, at min/max pts

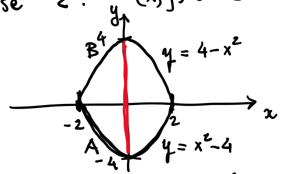
$\in \underline{\underline{\text{Int } D}} \quad \Rightarrow \quad \nabla f = \vec{0}$

$$\Leftrightarrow \begin{cases} 2x(1-y) = 0 \\ -x^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ 0 = 0 \end{cases} \Rightarrow (0,y),\ y \in \mathbb{R}$$

$\underline{(0,y)} \in \text{Int } D \Leftrightarrow \boxed{-4 < y < 4}$

Case 2 : $(x,y) \in \partial D = A \cup B$



where $A = \{ \underline{(x, x^2 - 4)} : \underline{-2 \leq x \leq 2} \}$

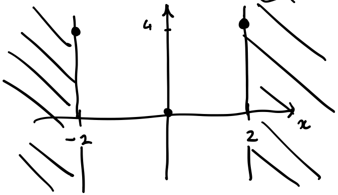
$B = \{ (x, 4 - x^2) : -2 \leq x \leq 2 \}$

On A : $f = f(x, x^2 - 4)$

$= x^2 (1 - (x^2 - 4))$

$= x^2 (5 - x^2) =: g(x)$

$x \in [-2, 2]$



$g(\pm 2) = 4$

$g(0) = 0$

$g'(x) = 2x(5 - x^2)$
$+ x^2(-2x)$

$$= 2x\left(5 - x^2 - x^2\right)$$

$$\underline{= 2\textcircled{x}\left(5 - 2x^2\right)} \geq 0$$

| | $-2$ | | $-\sqrt{5/2}$ | | $0$ | | $\sqrt{5/2}$ | | $2$ |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | $-$ | | $-$ | | $+$ | | $+$ | |
| $5-2x^2$ | | $-$ | | $+$ | | $+$ | | $-$ | |
| $g'$ | | $+$ | | $-$ | | $+$ | | $-$ | |
| $g$ | | ↗ | | ↘ | | ↗ | | ↘ | |

$$5 - 2x^2 \geq 0$$

$$x^2 \leq 5/2$$

$$-\sqrt{5/2} \leq x \leq \sqrt{5/2}$$

min $g$ $\quad x = \pm 2$ $\quad \Rightarrow \min\limits_{A} f \quad (\pm 2, 0)$

$\underline{x = 0}$ $\quad (0, -4)$

max $g$ $\quad x = \pm\sqrt{\frac{5}{2}} \Rightarrow \max\limits_{A} f \quad \left(\pm\sqrt{\frac{5}{2}}, \frac{5}{2} - 4\right)$

$$\underline{= \left(\pm\sqrt{\frac{5}{2}}, -\frac{3}{2}\right)}$$

On B: $\quad f(x, 4 - x^2) = x^2\left(1 - (4 - x^2)\right)$

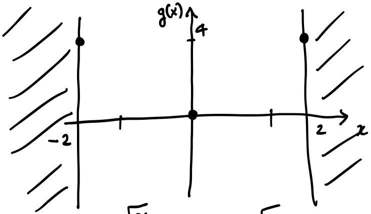
$$= x^2\left(x^2 - 3\right) =: g(x)$$

$$x \in [-2, 2].$$

$g(0) = 0, \quad g(\pm 2) = 4 \qquad g' = 2x(x^2 - 3)$

$$+ x^2 \cdot 2x$$

$$= 2x(2x^2 - 3)$$



$$2x^2 - 3 \geq 0$$

$$\Updownarrow$$

$$x^2 \geq 3/2$$

$$\Updownarrow$$

$$x \leq -\sqrt{3/2} \vee x \geq \sqrt{\frac{3}{2}}$$

| | $-2$ | | $-\sqrt{3/2}$ | | $0$ | | $\sqrt{3/2}$ | | $2$ |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | | $-$ | | $-$ | | $+$ | | $+$ | |
| $2x^2-3$ | | $+$ | | $-$ | | $-$ | | $+$ | |
| $g'$ | | $-$ | | $+$ | | $-$ | | $+$ | |
| $g$ | | ↘ | | ↗ | | ↘ | | ↗ | |

min $g$ for $\quad x = \pm\sqrt{\frac{3}{2}} \Rightarrow \min\limits_{B} f$ at $\left(\pm\sqrt{\frac{3}{2}}, 4 - \frac{3}{2}\right)$

$$= \left(\pm\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right)$$

max $g$ for $x = \pm 2$, $x = 0$ $\Rightarrow$ max $f$ at $(\pm 2, 0)$, $(0, 4)$

B

Conclusion: $f(x,y) = x^2(1-y)$

about mins, possible min pts are

- $(0,y)$ $-4<y<4$ $\xrightarrow{f}$ $= \boxed{0}$
- $(\pm 2, 0)$, $\longrightarrow$ $= \boxed{4}$
  $(0,-4)$ $\longrightarrow$ $= \boxed{0}$
- $\left(\pm\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ $\longrightarrow$ $= \frac{3}{2}\cdot\left(1-\frac{5}{2}\right) = \boxed{-\frac{9}{4}}$

$\Rightarrow$ global min pts are $\left(\pm\sqrt{\frac{3}{2}}, 5/2\right)$

About max, possible max pts are

- $(0,y)$, $-4<y<4$ $\xrightarrow{f}$ $\boxed{0}$
- $\left(\pm\sqrt{\frac{5}{2}}, -\frac{3}{2}\right) \leftarrow$ $\longrightarrow$ $\frac{5}{2}\cdot\left(1+\frac{3}{2}\right) = \boxed{\frac{25}{4}} \leftarrow$
- $(\pm 2, 0)$ $\longrightarrow$ $\boxed{4}$
  $(0, 4)$ $\longrightarrow$ $\boxed{0}$

$\left(\pm\sqrt{\frac{5}{2}}, -\frac{3}{2}\right)$ are the global max pts on D.

Q3: $f(D)$ = interval $= \left[-9/4, \frac{25}{4}\right]$ ■

$\mathcal{C}^{\nearrow}$ $\uparrow$ is connected

Do Ex 2.6.7