Friday, 3 March 2023 12:09

Unconstrained Opt. Pbs

Ex 2.6.9

$$f(x,y) := (x^2+y^2)^2 - 3x^2y \qquad (x,y)\in\mathbb{R}^2$$

Q1 Compute $\lim\limits_{(x,y)\to\infty_2} f$

Q2 Det st. pts of f

Q3. min/max for f on $\mathbb{R}^2$

Q4. $f(\mathbb{R}^2)$

Sol: Q1. $f(x,0) = (x^2)^2 = x^4 \xrightarrow[(x,0)\to\infty_2]{} +\infty$

$$f(x,y) \underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta\\ x^2+y^2=\rho^2}}{=} (\rho^2)^2 - 3(\underline{\rho}\cos\theta)^2\underline{\rho}\sin\theta$$

$$= \underset{\substack{\downarrow\\ +\infty}}{\rho^4} - 3\underset{\substack{\downarrow\\ +\infty}}{\rho^3}\,\underbrace{(\cos\theta)^2\sin\theta}_{-1\le \qquad \le 1}$$

$$(x,y)\to\infty_2 \Leftrightarrow \rho\to+\infty$$

$$\geq \rho^4 - 3\rho^3 \xrightarrow[\rho\to+\infty]{} +\infty$$

$$\Rightarrow\ \exists \lim_{(x,y)\to\infty_2} f = +\infty.$$

Q2.

$$\nabla f = (\partial_x f, \partial_y f)$$

$$f = \underbrace{(x^2+y^2)^2} - 3x^2y$$

$$\partial_x f = 2(x^2+y^2)2x - 3\cdot 2xy$$

$$= 4\textcircled{x}(x^2+y^2) - 6\textcircled{x}y$$

$$\partial_y f = 2(x^2+y^2)2y - 3x^2$$

$$= 4y\,(x^2+y^2) - 3x^2.$$

$(\partial_x f, \partial_y f \in \mathscr{C}(\mathbb{R}^2)) \Rightarrow$ (diff test) f is diff on $\mathbb{R}^2$)

(x,y) is st point $\Leftrightarrow \nabla f(x,y) = \vec{0}$

$$\Leftrightarrow \begin{cases} \boxed{x\left(4(x^2+y^2) - 6y\right) = 0} \\ 4y(x^2+y^2) - 3x^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ 4y^3 = 0 \end{cases} \quad \vee \quad \begin{cases} \boxed{4(x^2+y^2) - 6y = 0} \\ \quad // \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\boxed{(0,0)} \qquad\qquad \begin{cases} x^2+y^2 = \frac{3}{2}y \\ \underbrace{\overset{2}{\cancel{4}}y\cdot\cancel{\tfrac{3}{2}}y - \cancel{3}x^2}_{} = 0 \end{cases}$$

$$2y^2 - x^2 = 0$$

$$(\sqrt{2}y - x)(\sqrt{2}y + x) = 0$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$\Leftrightarrow \begin{cases} x^2+y^2 = \frac{3}{2}y \\ \boxed{(\sqrt{2}y - x)(\sqrt{2}y + x) = 0} \end{cases}$$

$$\Leftrightarrow \begin{cases} \sqrt{2}y - x = 0 \\ x^2+y^2 = \frac{3}{2}y \end{cases} \quad \vee \quad \begin{cases} \sqrt{2}y + x = 0 \\ x^2+y^2 = \frac{3}{2}y \end{cases}$$

$$\Leftrightarrow \begin{cases} x = \sqrt{2}y \\ \underbrace{2y^2+y^2}_{} = \frac{3}{2}y \end{cases} \quad \vee \quad \begin{cases} x = -\sqrt{2}y \\ 2y^2+y^2 = \frac{3}{2}y \end{cases}$$

$$2\cancel{3}y^2 = \frac{\cancel{3}}{\cancel{2}}y \qquad\qquad 2\cancel{3}y^2 = \frac{\cancel{3}}{\cancel{2}}y$$

$$\Leftrightarrow \begin{cases} x = \sqrt{2}\,y \\ y(2y-1) = 0 \end{cases} \vee \begin{cases} x = -\sqrt{2}\,y \\ y(2y-1) = 0 \end{cases}$$

$$\overset{\checkmark}{\boxed{\begin{matrix} y = 0 \\ x = 0 \end{matrix}}} \begin{matrix} \vee \\ \vee \end{matrix} \boxed{\begin{matrix} y = \frac{1}{2} \\ x = \frac{\sqrt{2}}{2} \end{matrix}} \quad \vee \quad \boxed{\begin{matrix} y = 0 \\ x = 0 \end{matrix}} \vee \begin{matrix} y = \frac{1}{2} \\ x = -\frac{\sqrt{2}}{2} \end{matrix}$$

Conclusion: st. pts are $(0,0), \left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right),$
$\left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

Q3. We have to det min/max of f on $\underbrace{\mathbb{R}^2}_{\text{closed and unbounded}}$ (if any)

and since $f \in \mathcal{C}(\mathbb{R}^2)$, by Q1

$$\lim_{\infty_2} f = +\infty$$

$\Rightarrow \nexists \max\limits_{\mathbb{R}^2} f, \quad \exists \min\limits_{\mathbb{R}^2} f.$

Let's find min pt: if $(x,y) \in \mathbb{R}^2$ is a min pt for $f \Rightarrow$

- $(x,y) \in \operatorname{Int} \mathbb{R}^2 = \mathbb{R}^2 \Rightarrow$ (Fermat)
- $(x,y) \in \mathbb{R}^2 \setminus \operatorname{Int} \mathbb{R}^2 = \emptyset$

$$\Downarrow$$
$$\nabla f = \vec{0}$$
$$\Downarrow$$

(x,y) is among $(0,0), \left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

$f(0,0) = \boxed{0}$

$$f\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \left(\frac{1}{2} + \frac{1}{4}\right)^2 - 3\,\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{9}{16} - \frac{3}{4} = \boxed{\frac{-3}{16}}$$

$\Rightarrow \left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ are mins for f.

Q4. $\underset{\substack{\uparrow \\ \text{connected}}}{f(\mathbb{R}^2)} = \text{interval} = \left[-\frac{3}{16}, +\infty\right[$.

Ex 2.6.10. $f(x,y,z) = (x^2+y^2+z^2)^2 - xyz$

Q1 $\lim\limits_{\infty_3} f$

Q2 St. pt

Q3 min/max f on $\mathbb{R}^3$

Q4 $f(\mathbb{R}^3)$.

Sol: Q1 $f \underset{\substack{x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi}}{=} (\rho^2)^2 - \rho^3 \underset{-1 \leq \quad \leq 1}{C(\theta,\varphi)}$

$$\geqslant \rho^4 - \rho^3 \xrightarrow[\rho = \|(x,y,z)\| \to +\infty]{} +\infty$$

$$\Updownarrow$$

$$(x,y,z) \to \infty_3$$

$\Rightarrow \quad \exists \lim\limits_{\infty_3} f = +\infty.$

Q2. We solve $\nabla f = \vec{0}$

$$\begin{cases} \partial_x f = \boxed{\cancel{2}4(x^2+y^2+z^2)\ x - yz = 0} \\ \partial_y f = \underbrace{\cancel{2}4(x^2+y^2+z^2)}\cancel{2}y - xz = 0 \\ \partial_z f = \cancel{2}4(x^2+y^2+z^2)\cancel{2}z - xy = 0 \end{cases}$$

$$\Updownarrow$$

$$x \neq 0 \begin{cases} 4(x^2+y^2+z^2) = \boxed{\frac{yz}{x}} \\ \frac{yz}{x}y - xz = 0 \end{cases} \quad \vee \quad \begin{cases} x = 0 \\ -\boxed{yz = 0} \\ \cancel{4}(y^2+z^2)y = 0 \\ \cancel{4}(y^2+z^2)z = 0 \end{cases}$$

$$\frac{yz}{x}z - xy = 0$$

$$\Updownarrow$$

$$\begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ \boxed{\textcircled{z}(y-x)(y+x)=0} \\ y(z-x)(z+x)=0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x \neq 0 \\ 4(x^2+y^2\cancel{+z^2}) = \frac{y\cancel{z}}{\cancel{x}}\,0 \\ z=0 \quad \Rightarrow \quad x=0,\ y=0 \end{cases} \lor \begin{cases} \boxed{x \neq 0} \\ y=x \\ 4(2x^2+z^2)=z \\ \cancel{x}(z-x)(z+x)=0 \end{cases} \lor \begin{cases} x \neq 0 \\ y=-x \\ 4(2x^2+z^2)=-z \\ \cancel{-x}(z-x)(z+x)=0 \end{cases}$$

$$\Downarrow$$

NO SOLS

$$\begin{cases} \boxed{x \neq 0} \\ y=x \\ z=x \\ 4\cdot 3x^{\cancel{2}} = \cancel{x}1 \end{cases} \lor \begin{cases} x \neq 0 \\ y=x \\ z=-x \\ 4\cdot 3x^{\cancel{2}} = -\cancel{x}1 \end{cases} \qquad \begin{cases} x \neq 0 \\ y=-x \\ z=x \\ 4\cdot 3x^{\cancel{2}} = -\cancel{x}1 \end{cases} \lor \begin{cases} x \neq 0 \\ y=-x \\ z=-x \\ 4\cdot 3x^{\cancel{2}} = \cancel{x}1 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x = 1/12 \\ y = 1/12 \\ z = 1/12 \end{cases} \lor \begin{cases} x = -1/12 \\ y = -1/12 \\ z = +1/12 \end{cases} \lor \begin{cases} x = -1/12 \\ y = +1/12 \\ z = -1/12 \end{cases} \lor \begin{cases} x = 1/12 \\ y = -1/12 \\ z = -1/12 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x=0 \\ y=0 \\ z^3=0 \end{cases} \lor \begin{cases} x=0 \\ z=0 \\ y^3=0 \end{cases}$$

$$\boxed{(0,0,0)}$$

$\lor\ (\ast)$

Q3. By Q1 $\nexists \max_{\mathbb{R}^3} f$

$f \in \mathcal{C}(\mathbb{R}^3)$, $\lim_{\infty_3} f = +\infty$, $\mathbb{R}^3$ closed and unbounded

$\Rightarrow \exists \min_{\mathbb{R}^3} f$.

Let's find the min pts.

If $(x,y,z) \in \mathbb{R}^3 = \operatorname{Int}\mathbb{R}^3$ is a min pt for f in this case $\Rightarrow$ (F.) $\nabla f = \vec{0}$

$\Rightarrow (x,y,z)$ is one of 5 pts

found in Q2.

$$f(0,0,0) = \boxed{0}$$

in the other cases $f = \left(3\left(\frac{1}{12}\right)^2\right)^2 - \frac{1}{12^3}$

$$= \frac{9}{12^4} - \frac{1}{12^3}$$

$$= \frac{-3}{12^4} \quad \boxed{< 0}$$

$\Rightarrow$ the min pts are $(*)$

Q4. $f(\mathbb{R}^3) = \left[-\frac{3}{12^4}, +\infty\right[$. ■

Do Ex.

2.6.7

2.6.8 i), iv)

2.6.12

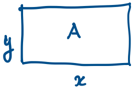
Constrained Optimization

We consider now optimization pbs where we have to find

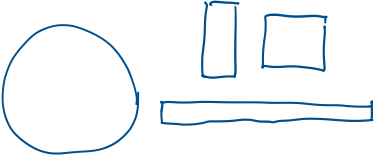
$$\min_{D}/\max_{D} f(\vec{x})$$

$$D = \{\vec{x} \in \mathbb{R}^d : g(\vec{x}) = 0\}$$

Ex 1:



Fixed the perimeter P, det all poss. rects s.t. area is max.



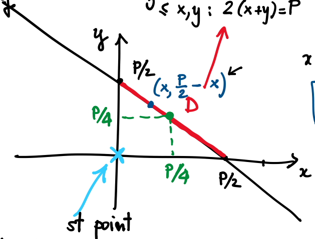
given

$\boxed{P} = 2(x+y)$

$A = xy$

Pb: $\max \; xy$

$0 \leq x,y : 2(x+y) = P$

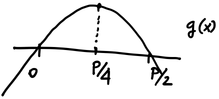


$x + y = P/2$

$\boxed{y = P/2 - x}$

$f(x,y) = x \cdot y$

$f(x, \frac{P}{2} - x) = x\left(\frac{P}{2} - x\right) = g(x)$



$x = P/4$ max for $g \Rightarrow (P/4, P/2 - P/4)$

$\parallel$

$(P/4, P/4)$ ∎

Rmk: In the prev ex $f = xy$

$\nabla f = (y, x) = \vec{0} \Leftrightarrow (0,0)$

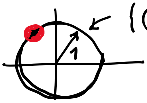
$\Rightarrow \nabla f \neq 0 \quad \forall (x,y) \in D$ ∎

Let's try to learn something from the example: we had to solve max/min $f(x,y)$

$$\boxed{(x,y):\ g(x,y)=0}$$

Example $f(x,y) = xy \qquad g(x,y) = 2(x+y)-P$

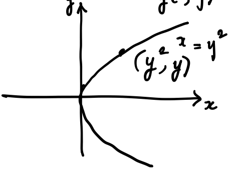
Examples:

- 

$\{(x,y) : x^2+y^2=1\}$

$$\underbrace{x^2+y^2-1}_{g(x,y)}=0$$

- $\{(x,y) : \underbrace{x-y^2}_{g(x,y)} = 0\}$



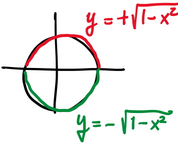
$x = y^2$

- $\{(x,y) : x^2+y^2+1 = 0\} = \phi$

Typically (apart exceptional cases), a set

$$\{(x,y)\in\mathbb{R}^2 : g(x,y)=0\}$$

is a "line" in $\mathbb{R}^2$ with no interior points.



$x^2+y^2=1$

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2}$

In general, we may expect that

$$(x,y) \;:\; g(x,y)=0 \Leftrightarrow \boxed{y=\varphi(x)} \text{ or } x=\varphi(y)$$

$$\Rightarrow \quad \underset{g(x,y)=0}{\min/\max} f(x,y) = \underset{x}{\min/\max} \underset{\substack{\parallel \\ h(x)}}{\boxed{f(x,\varphi(x))}}$$

If x_0 is min/max for $h(x)$

(for ex. x_0 be a min) $\Rightarrow$ $\boxed{h'(x)=0}$

$$0 = h'(x) = \frac{d}{dx}\left[f(x,\varphi(x))\right]$$

$$= \boxed{\partial_x f(x,\varphi(x)) + \partial_y f(x,\varphi(x))\,\varphi'(x)}$$

$$\ulcorner \frac{d}{dt} f(\gamma_1(t),\gamma_2(t)) \underset{\substack{\uparrow \\ \text{total deriv}}}{=} \nabla f \cdot (\gamma_1'(t), \gamma_2'(t))$$

$$= \partial_x f \cdot \gamma_1'(t) + \partial_y f \cdot \gamma_2' =$$

$\gamma_1(t) = t$

$\gamma_2(t) = \varphi(t)$ $\qquad$ $f(\gamma_1(t),\gamma_2(t)) = f(t,\varphi(t))$

$\gamma_1' = 1$

$\gamma_2' = \varphi'$ $\qquad$ $= \partial_x f \cdot 1 + \partial_y f \cdot \varphi'$ $\lrcorner$

$$\Rightarrow \quad h'(x) = 0 \quad \text{means}$$

$$\boxed{\partial_x f + \partial_y f\, \varphi' = 0} \quad (1)$$

What is φ'? Recall that

$$g(x,y)=0 \Leftrightarrow y=\varphi(x)$$

$$\Downarrow$$

$$g(x, \varphi(x)) = 0 \qquad \forall x$$

$$\Downarrow$$

$$\frac{d}{dx}\, g(x, \varphi(x)) = 0$$

$$\Downarrow$$

$$\partial_x g + \partial_y g\, \varphi' = 0$$

IF $\boxed{\partial_y g \neq 0}$

$$\varphi'(x) = -\frac{\partial_x g}{\partial_y g}.$$

We plug this into (1) and we get

$$\partial_x f + \partial_y f \left(- \frac{\partial_x g}{\partial_y g} \right) = 0$$

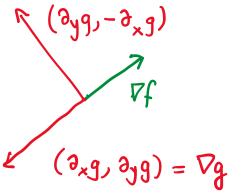
$$\Updownarrow$$

$$\partial_x f\, \partial_y g - \partial_y f\, \partial_x g = 0$$

$$(\partial_x f, \partial_y f) \cdot (\partial_y g, -\partial_x g) = 0$$

$$\Updownarrow$$

$$\nabla f \perp (\partial_y g, -\partial_x g)$$


$(\partial_y g, -\partial_x g)$
∇f
$(\partial_x g, \partial_y g) = \nabla g$

$$\Rightarrow \boxed{\exists \lambda : \nabla f = \lambda \nabla g}$$

Conclusion: If (x_0, y_0) is a min/max for $f = f(x,y)$ on $D = \{g(x,y) = 0\}$ and $\partial_y g(x_0, y_0) \neq 0$ then

$$\exists \lambda : \quad \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

In the same way, one proves that If (x_0, y_0) is min/max pt for f on $D = \{g(x,y) = 0\}$ and $\partial_x g(x_0, y_0) \neq 0$ then

$$\exists \lambda : \quad \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Combining the two statements we obtain

Thm (Lagrange)

At (x_0, y_0) min/max pt for $f = f(x,y)$ on $D = \{g(x,y) = 0\}$, if $\nabla g(x_0, y_0) \neq \vec{0}$ then

$$\exists \lambda : \quad \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Example Let's apply this result to solve

$$\max_{(x,y) :\, x \geq 0,\, y \geq 0,\, \boxed{x + y = P/2}} xy$$

Here $\qquad\qquad x + y - \frac{P}{2} = 0$

$$f(x,y) = xy$$

$$g(x,y) = x + y - P/2 .$$

At max pt (x,y), $\nabla g = (1,1) \neq \vec{0}$

we must have

$$\exists \lambda : \quad \nabla f = \lambda \nabla g$$

$\nabla f = (y, x)$

$$\nabla f = \lambda \nabla g \iff (y, x) = \lambda(1,1)$$

$$\iff \begin{cases} y = \lambda \\ x = \lambda \end{cases} \Rightarrow (x,y) = (\lambda, \lambda)$$

Imposing $(x,y) \in D$, $x + y = \frac{p}{2} \Rightarrow$

$$2\lambda = \frac{p}{2}$$

$$\lambda = p/4$$

$$\Rightarrow (x,y) = \left(p/4, p/4\right) \quad \blacksquare$$

Ex. 2.4.2 Solve

$$\min/\max \; (x+y)$$
$$x^2+y^2 = 1$$

Sol: Here $f(x,y) = x+y$

$$D = \{ \underbrace{x^2+y^2-1}_{g(x,y)} = 0 \}$$

Existence: $f \in \mathcal{C}(\mathbb{R}^2)$, D is closed (because defd by an equality with $g \in \mathcal{C}$) and bounded ($\|(x,y)\| = 1 \; \forall (x,y) \in D$)

$\Downarrow$

(Weierstrass) f has both min/max on D.

Determination: if $(x,y) \in D$ is min/max for f, we aim to apply Lagr. Let's check first if $\nabla g \neq \vec{0}$

Now

$$\nabla g = 0 \iff (2x, 2y) = (0,0) \iff (x,y) = \underset{\notin D}{(0,0)}$$

$$\Rightarrow \quad \nabla g \neq \vec{0} \quad \text{on } D$$

$$\Rightarrow \quad \boxed{\nabla f = \lambda \nabla g}$$

$$\iff \quad (1,1) = \lambda\,(2x, 2y)$$

$$\iff \begin{cases} 2\lambda x = 1 \\ 2\lambda y = 1 \end{cases} \Rightarrow \begin{cases} x = 1/2\lambda \\ y = 1/2\lambda \end{cases}$$

$$\Rightarrow \quad \left(\frac{1}{2\lambda}, \frac{1}{2\lambda}\right) \in D \iff \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\iff \frac{1}{2\lambda^2} = 1$$

$$\iff \lambda^2 = 1/2$$

$$\iff \lambda = \pm 1/\sqrt{2}$$

$$\begin{array}{ll} \lambda = 1/\sqrt{2} & \rightsquigarrow \\ & \text{max} \nearrow \\ \lambda = -1/\sqrt{2} & \rightsquigarrow \end{array} \boxed{\begin{array}{l} \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \xrightarrow{f} \sqrt{2} \\ \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \xrightarrow{f} -\sqrt{2} \end{array}}$$

$\nearrow$ min.

Do Ex 2.6.15 ii), iii).