Monday, 6 March 2023 12:01

Lagrange multiplier thm

Pb: $$\boxed{\begin{array}{c} \min/\max \ f(x,y) \ . \\ (x,y) : g(x,y)=0 \end{array}}$$

If (x,y) is a min/max pt for f on $D = \{g(x,y)=0\}$ and $\nabla g(x,y) \neq \vec{0}$

$$\Rightarrow \quad \nabla f(x,y) = \lambda \nabla g(x,y).$$

Ex 2.6.15 ii)

$\underset{D}{\min/\max} f$

$f(x,y) = 2x^2+y^2-x$

$D = \{\underline{x^2+y^2=1}\}$

Sol: $f \in \mathscr{C}$, D is closed $\left(D=\{g=0\} \quad g(x,y) = \underbrace{x^2+y^2-1}_{\in \mathscr{C}} \right)$

and bded $\left(\|(x,y)\| = 1 \quad \forall (x,y) \in D \right)$

$(\Rightarrow D$ is compact$) \Rightarrow$ (W. thm)

$\exists$ min/max f.

D

To find min/max pts on D we apply the Lagrange thm: to check the technical assumption

$$\nabla g \neq \vec{0} \quad \text{at min/max pts}$$

let's see where $\nabla g = \vec{0}$:

$$g = x^2 + y^2 - 1$$

$$\nabla g = (2x, 2y) = \vec{0} \iff (x,y) = (0,0)$$

But since $(0,0) \notin D \Rightarrow \nabla g \neq \vec{0}$ on D

$\Rightarrow$ wherever min/max pts are on D, the assumptions of Lagrange thm are verified

$\Rightarrow$ at min/max pt (x,y) we have

$$\boxed{\underline{\nabla f(x,y)} = \lambda \, \underline{\nabla g(x,y)}.} \quad (*)$$

$$f = \underline{2x^2 + y^2 - x}$$

$$\nabla f = (4x - 1, 2y)$$

$$(*) \iff (4x - 1, 2y) = \lambda (2x, 2y)$$

$$\Leftrightarrow \quad \cdot \begin{cases} 4x - 1 = \lambda 2x \\ 2y = \lambda 2y \end{cases}$$

$$\Leftrightarrow \begin{cases} 2y(1-\lambda) = 0 \\ \cancel{2}(2-\lambda)x = 1/2 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 0 \\ (2-\lambda)x = 1/2 \end{cases} \quad \vee \quad \begin{cases} \lambda = 1 \\ x = 1/2 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = 0 \\ x = \dfrac{1}{2(2-\lambda)} \end{cases} \qquad \underline{\left(\tfrac{1}{2}, y\right)} \qquad \lambda = 1$$

$$\underline{\left(\frac{1}{2(2-\lambda)}, 0\right)}$$

Now $$\left(\frac{1}{2(\underline{2-\lambda})}, 0\right) \in D \Leftrightarrow \frac{1}{4(2-\lambda)^2} + \cancel{0} = 1$$

$$\Leftrightarrow 1 = 4(2-\lambda)^2$$

$$\Leftrightarrow (\lambda - 2)^2 = 1/4$$

$$\Leftrightarrow \lambda - 2 = \pm 1/2$$

$$\Leftrightarrow \lambda = 2 \pm 1/2 = 5/2, 3/2$$

$$\boxed{(1,0),\ (-1,0)}$$

$$\left(\frac{1}{2}, y\right) \in D \iff \frac{1}{4} + y^2 = 1 \Rightarrow y^2 = \frac{3}{4}$$

$$\Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \boxed{\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)}$$

$f = 2x^2 + y^2 - x$,

$$f(1,0) = \boxed{1}$$

$$f(-1,0) = \boxed{3}$$

$$f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \cancel{\cancel{2}\frac{1}{\cancel{4}\,2}} + \frac{3}{4} \cancel{-\frac{1}{2}}$$

$$= \boxed{\frac{3}{4}}$$

Conclusion: $(-1,0)$ is max pt

$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ are min pts. ∎

Rmk: $\nabla f(-1,0) = (-5,0)$

$$\nabla f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = (1, \pm\sqrt{3})$$

∎

Rmk: Can we solve

$\boxed{\nabla f = \lambda \nabla g}$ (since $\nabla g \neq \vec{0}$)

without det λ ?

$\Leftrightarrow$ dim vect space generated by $\nabla f, \nabla g = 1$

$$\Leftrightarrow \quad \underset{2\times 2}{\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix}} = 1$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \quad \Leftrightarrow$$

$$\boxed{\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0}$$

So, for instance, in prev ex.

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0$$

$$\Leftrightarrow \quad 0 = \det \begin{bmatrix} 4x-1 & 2y \\ 2x & 2y \end{bmatrix}$$

$$0 = 2y(4x-1) - 4xy$$

$$= 2y(4x-1-2x)$$

$$0 = \cancel{2}y(2x-1)$$

$$\Leftrightarrow \quad y = 0 \quad \lor \quad 2x-1=0$$

$$\Updownarrow$$

$$(x,0) \; x\in\mathbb{R} \quad \lor \quad \left(\tfrac{1}{2}, y\right) \; y\in\mathbb{R}.$$

$$(x,0)\in D \quad \Leftrightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

$$(\pm 1, 0)$$

$$\left(\tfrac{1}{2}, y\right)\in D \quad \Leftrightarrow \quad \frac{1}{4} + y^2 = 1 \Leftrightarrow \quad y = \pm\frac{\sqrt{3}}{2}$$

$$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right).$$

■

2.6.15 iv)

$$f = xy$$

$$D = \{x^2+y^2+xy-1=0\}$$

min/max f

D

Sol: Existence: $f\in\mathcal{C}$, $D=\{g=0\}$,

$g = x^2+y^2+xy-1$, $g\in\mathcal{C} \Rightarrow D$ is closed

Is it bounded?

$$(x,y) \in D \iff x^2 + y^2 + xy - 1 = 0$$

$$\rho = \|(x,y)\| = \sqrt{x^2+y^2}$$

$$\underset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{\iff} \quad \rho^2 + \rho^2 \cos\theta \sin\theta = 1$$

$$\iff \quad \rho^2 (1 + \cos\theta \sin\theta) = 1$$

$$\iff \quad \rho^2 = \frac{1}{1 + \cos\theta \sin\theta} \leq M$$

$$= \frac{1}{\underbrace{1 + \frac{1}{2}\sin(2\theta)}_{\frac{1}{2} \leq \quad \leq \frac{3}{2}}} \leq 2$$

$\Rightarrow$ D is bounded

$\Rightarrow$ $\exists$ min/max pts for f on D

To find them, we apply L. thm.

We need to check if $\nabla g \neq \vec{0}$ at min/max pts

We check when $\nabla g = \vec{0}$

$$\nabla g = (2x+y, 2y+x) = \vec{0} \iff \begin{cases} 2x+y=0 \\ 2y+x=0 \end{cases}$$

$$g = x^2+y^2+xy-1 = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

$\iff (x,y) = (0,0) \in D$? NO

$\Rightarrow \nabla g \neq \vec{0}$ on D

Therefore, by L. thm, at min/max pt (x,y) for f on D we must have

$$\nabla f = \lambda \nabla g \iff \det\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0$$

$f = xy$

$$\iff \det\begin{bmatrix} y & x \\ 2x+y & 2y+x \end{bmatrix} = 0$$

$$\iff y(2y+x) - x(2x+y) = 0$$

$$\iff \cancel{2}y^2 + \cancel{xy} - \cancel{2}x^2 - \cancel{xy} = 0$$

$$\iff y^2 - x^2 = 0 \iff \boxed{(y-x)(y+x)=0}$$

$$\Leftrightarrow \quad y = x \quad \vee \quad y = -x$$

$$\boxed{(x,x),\ x\in\mathbb{R} \ \vee \ (x,-x),\ x\in\mathbb{R}}$$

Now

$$(x,x) \in D \Leftrightarrow x^2 + x^2 + x^2 - 1 = 0$$

$$\Leftrightarrow x^2 = 1/3$$

$$\Leftrightarrow x = \pm \frac{1}{\sqrt{3}}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$(x,-x) \in D \Leftrightarrow x^2 + \cancel{x^2} - \cancel{x^2} - 1 = 0$$

$$\Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$$

$$(1,-1), (-1,1)$$

Conclusion:

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \boxed{\frac{1}{3}}$$

$$f(1,-1) = f(-1,1) = \boxed{-1}$$

$\Rightarrow$ $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are max pts

$(1,-1), (-1,1)$ are min pts.