Tuesday, 7 March 2023 12:00

Let's consider now the pb

$$\min/\max_{\vec{x}\in\mathbb{R}^d \,:\, g(\vec{x})=0} f(\vec{x})$$

(we discussed

$$\min/\max_{(x,y)\in\mathbb{R}^2 \,:\, g(x,y)=0} f(x,y)$$

)

Thm (Lagrange)

Let $\vec{x}_0$ be a min/max pt for f on $D = \{g=0\}$. If $\boxed{\nabla g(\vec{x}_0) \neq 0}$ then

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0).$$

Rmk: If $d = 2$

2x2

$$\nabla f = \lambda \nabla g \iff \operatorname{rk}\boxed{\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix}} < 2$$

$$\iff \det\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} = 0$$

If $d = 3$

2x3 matrix

$$\nabla f = \lambda \nabla g \iff \operatorname{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2$$

$$\begin{bmatrix}\partial_x f & \partial_y f & \partial_z f\\ \partial_x g & \partial_y g & \partial_z g\end{bmatrix}$$

all 2x2 sub-dets $= 0$

$$\Leftrightarrow \begin{cases} \det \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{bmatrix} = 0 \\ \det \begin{bmatrix} \partial_x f & \partial_z f \\ \partial_x g & \partial_z g \end{bmatrix} = 0 \\ \det \begin{bmatrix} \partial_y f & \partial_z f \\ \partial_y g & \partial_z g \end{bmatrix} = 0 \end{cases}$$

Ex 2.6.16

$\underset{D}{\min/\max} f$

i) $f = x - 2y + 2z \qquad D = \{(x,y,z) : x^2+y^2+z^2 = 9\}$

Sol: $f \in \mathcal{C}$, D is closed & bded

$\Rightarrow$ (W.) $\exists$ min/max for f on D

To det min/max pts, we apply L. thm:

Notice first that $\nabla g = \vec{0} \Leftrightarrow (2x, 2y, 2z)$

$g = x^2+y^2+z^2-9$

$= (0,0,0)$

$\Leftrightarrow (x,y,z) = \vec{0} \notin D$

$\Rightarrow \nabla g \neq \vec{0}$ on D.

$\Rightarrow$ at min/max pts $(x,y,z) \underline{\underline{\in D}}$

$$\boxed{\nabla f = \lambda \nabla g}$$

$$\Leftrightarrow \quad \operatorname{rk} \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 1 & -2 & 2 \\ 2x & 2y & 2z \end{bmatrix} < 2$$

$$\Leftrightarrow \begin{cases} \det\begin{bmatrix} 1 & -2 \\ 2x & 2y \end{bmatrix} = 0 \\ \det\begin{bmatrix} 1 & 2 \\ 2x & 2z \end{bmatrix} = 0 \\ \det\begin{bmatrix} -2 & 2 \\ 2y & 2z \end{bmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} \cancel{2}y + \overset{2}{\cancel{4}}x = 0 \\ \cancel{2}z - \overset{2}{\cancel{4}}x = 0 \\ +\cancel{4}z + \cancel{4}y = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = -2x \\ z = 2x \\ \cancel{0 = 0} \end{cases} \Rightarrow (x, -2x, 2x)$$

$$x \in \mathbb{R}.$$

$$\underline{(x,-2x,2x)} \in D \Leftrightarrow x^2 + 4x^2 + 4x^2 = 9$$

$$\Leftrightarrow 9x^2 = 9$$

$$\Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$$

$$\begin{tikzcd} (1,-2,2), \arrow[d, "f=x-2y+2z"] & (-1,2,-2) \arrow[d, "f"] \\ 1-2(-2)+2\cdot 2 = \boxed{9} & -1-2\cdot 2+2(-2) = \boxed{-9} \end{tikzcd}$$

$\Rightarrow$ $(1,-2,2)$ is max pt for f on D

$(-1,2,-2)$ " min " " " " . ∎

Do 2.6.16 ii)

__Def:__ We say that $g = g(\vec{x})$ is a __submersion__ on D if

$\nabla g \neq \vec{0}$ on D

Ex 2.6.18

$\mathcal{M} = \{(x,y,z)\in\mathbb{R}^3 : z^2 = xy+1\}$

Q1. $\mathcal{M} \neq \emptyset$ $\quad \mathcal{M} = \{g=0\}$ where g is a submersion on $\mathcal{M}$. $\quad (x,y,0)\in\mathcal{M}$ $\Leftrightarrow xy=-1$ $\quad (x,-1/x,0)$

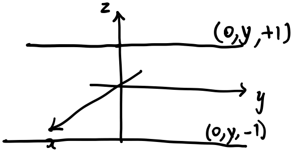
Q2. Is $\mathcal{M}$ compact?

Q3. Find pts of $\mathcal{M}$ (if any) at min/max distance to $\vec{0}$.

Q1. $(0,0,z)\in\mathcal{M} \Leftrightarrow z^2=1 \Rightarrow (0,0,\pm1)\in\mathcal{M}$

$\Rightarrow \mathcal{M}\neq\emptyset$

$(0,y,z)\in\mathcal{M} \Leftrightarrow z^2=1 \Rightarrow \boxed{\begin{array}{c}(0,y,\pm1)\in\mathcal{M}\\ \forall y\in\mathbb{R}\end{array}}$



$\mathcal{M} = \{g=0\} \qquad g = z^2 - xy - 1$

Goal: g is a submersion on $\mathcal{M}$

$\Leftrightarrow \nabla g \neq \vec{0}$ on $\mathcal{M}$.

Now: $\nabla g = \vec{0} \Leftrightarrow \underline{(-y,-x,2z)} = (0,0,0)$

$\Leftrightarrow (x,y,z) = (0,0,0) \notin \mathcal{M}$

because $0^2 = 0\cdot0+1$ false

$\Rightarrow \nabla g \neq \vec{0}$ on $\mathcal{M}$.

Q2. $\mathcal{M} = \{g = 0\}$, $g \in \mathcal{C} \Rightarrow \mathcal{M}$ is closed

Is $\mathcal{M}$ also bounded?

Since $(0, y, \pm 1) \in \mathcal{M} \quad \forall y \in \mathbb{R}$

$$\|(0, y, \pm 1)\| = \sqrt{y^2 + 1} \xrightarrow[y \to \pm\infty]{} +\infty$$

$\Rightarrow \mathcal{M}$ is unbounded.

Q3. We have to find pts $(x, y, z) \in \mathcal{M}$

s.t. $\|(x, y, z)\|$ is min/max

$$\boxed{f(x,y,z) := \overset{\|}{\sqrt{x^2 + y^2 + z^2}}}$$

Instead of min/max f we min/max

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$\min_{\mathcal{M}}/\max f.$$

$\nexists \max\limits_{\mathcal{M}} f \quad : \quad \underset{\in \mathcal{M}}{f(0, y, \pm 1)} = y^2 + 1 \xrightarrow[y \to \pm\infty]{} +\infty$

$\Rightarrow$ no global max is poss.

About min, we have $f \in \mathcal{C}$, $\mathcal{M}$ closed and unbded

and $$\lim_{(x,y,z) \to \infty} f = +\infty,$$

$\Rightarrow \exists \min\limits_{\mathcal{M}} f$. Let's find it: by Q1

$\Rightarrow \nabla g \neq \vec{0}$ on $\mathcal{M} \Rightarrow$

we can apply L. thm:

at min pt $\boxed{\nabla f = \lambda \nabla g} \Leftrightarrow$

$$2 > rk\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} = rk\begin{bmatrix}2x & 2y & 2z\\ -y & -x & 2z\end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \det\begin{bmatrix}2x & 2y\\ -y & -x\end{bmatrix} = 0\\ \det\begin{bmatrix}2x & 2z\\ -y & 2z\end{bmatrix} = 0\\ \det\begin{bmatrix}2y & 2z\\ -x & 2z\end{bmatrix} = 0\end{cases} \Leftrightarrow \begin{cases} -\cancel{2}x^2+\cancel{2}y^2=0\\ \overset{2}{\cancel{4}}xz+\cancel{2}yz=0\\ \overset{2}{\cancel{4}}yz+\cancel{2}xz=0\end{cases}$$

$$\Leftrightarrow \begin{cases} \boxed{(y-x)(y+x)=0}\\ z(2x+y)=0\\ z(2y+x)=0\end{cases}$$

$$\Leftrightarrow \begin{cases} y = x\\ \underbrace{xz}=0\end{cases} \vee \begin{cases} y=-x\\ zx=0\end{cases}$$

$$\begin{cases} y = x\\ x=0\end{cases} \quad \begin{cases} y=x\\ z=0\end{cases} \vee \begin{cases} y=-x\\ x=0\end{cases} \vee \begin{cases} y=-x\\ z=0\end{cases}$$

$$\underset{z\in\mathbb{R}}{(0,0,z)} \quad \underset{x\in\mathbb{R}}{\underbrace{(x,x,0)}} \quad \cancel{\underset{z\in\mathbb{R}}{(0,0,z)}} \quad \underset{x\in\mathbb{R}}{(x,-x,0)}$$

$(0,0,z)\in M \Leftrightarrow z^2=1 \Rightarrow \boxed{(0,0,\pm1)}$

$(x,x,0)\in M \Leftrightarrow 0=x^2+1$ no pts

$(x,-x,0) \in \mathcal{M} \iff 0 = -x^2+1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$$\boxed{(1,-1,0),\ (-1,1,0)}$$

$f(0,0,\pm 1) = \boxed{1}$

$f(1,-1,0) = f(-1,1,0) = 2$

$\Rightarrow$ Answ: pts of $\mathcal{M}$ at min dist to $\vec{0}$ are $(0,0,\pm 1)$! ∎

Do Ex 2.6.19 (*), 2.6.22

We extend the pb

$$\begin{aligned} &\min/\max \quad f(\vec{x}) \\ &g(\vec{x}) = 0 \end{aligned}$$

to

$$\begin{aligned} &\min/\max \quad f(\vec{x}) \\ &g_1(\vec{x}) = 0 \\ &g_2(\vec{x}) = 0 \\ &\vdots \\ &g_m(\vec{x}) = 0 \end{aligned}$$

Def: We say that $(g_1 \cdots g_m)$ are a submersion on D if

$$\nabla g_1, \nabla g_2, \cdots, \nabla g_m$$

are linearly indep on D

Rmk: $\nabla g_1, \ldots, \nabla g_m$ are lin indep $\Leftrightarrow$

$d > m$

$$\text{rk} \quad m \text{ lines} \left\{ \underbrace{\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \\ \vdots \\ \nabla g_m \end{bmatrix}}_{d \text{ cols}} \right. \qquad m \times d \text{ matrix}$$

$$\text{rk} \begin{bmatrix} \boxed{\nabla g_1} \\ | \\ \nabla g_m \end{bmatrix} = m$$

$$\Leftrightarrow \boxed{\begin{matrix} \partial_1 g_1 & \partial_2 g_1 & \partial_3 g_1 & - & \cdots & \partial_d g_1 \\ \partial_1 g_2 & \partial_2 g_2 & \partial_3 g_2 & - & - \cdots & \partial_d g_2 \\ \vdots & & & & & \\ \partial_1 g_m & \partial_2 g_m & - \sim \sim & & & \partial_d g_m \end{matrix}}$$

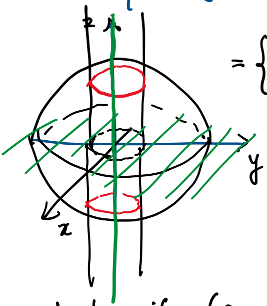
at least m out of d columns are linearly indep $\Leftrightarrow$

$\exists$ $m \times m$ submatrix of $\begin{bmatrix} \nabla g_1 \\ | \\ \nabla g_m \end{bmatrix}$ whose det $\neq 0$.

In other words, $(g_1 - g_m)$ is not a submersion if $\exists\ \vec{x}$ at which

all $m \times m$ submatrices of $\begin{bmatrix} \nabla g_1 \\ | \\ \nabla g_m \end{bmatrix}$ have det $= 0$

Example: $\mathcal{M} = \{ x^2 + y^2 + z^2 = 1, \ \underline{x^2 + y^2 = 1/4} \}$



$$= \left\{ \overbrace{x^2+y^2+z^2-1}^{g_1} = 0 \quad \underbrace{x^2+y^2-1/4}_{g_2} = 0 \right\}$$

Let's check if (g_1, g_2) is a submersion $\mathcal{M}$. $\Leftrightarrow$ $\nabla g_1, \nabla g_2$ are lin. indep on $\mathcal{M}$

$$\Leftrightarrow \quad \mathrm{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2.$$

Let's check when this $\underline{\text{is not}}$ true

$$\mathrm{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 2$$

$$\|$$

$$\mathrm{rk} \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \end{bmatrix} < 2$$

$$\Leftrightarrow \left\{ \begin{array}{ll} \det \begin{bmatrix} 2x & 2y \\ 2x & 2y \end{bmatrix} = 0 & 0 = 0 \\ \det \begin{bmatrix} 2x & 2z \\ 2x & 0 \end{bmatrix} = 0 & \cancel{-4}xz = 0 \\ \det \begin{bmatrix} 2y & 2z \\ 2y & 0 \end{bmatrix} = 0 & \cancel{-4}yz = 0 \end{array} \right.$$

$$\Leftrightarrow \begin{cases} \boxed{xz=0} \\ yz=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ \boxed{yz=0} \end{cases} \vee \begin{cases} z=0 \\ \cancel{0=0} \end{cases}$$

$$\boxed{\begin{matrix} (x,y,0) \\ x,y\in\mathbb{R} \end{matrix}}$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ z=0 \end{cases}$$

$$\boxed{\begin{matrix} (0,0,z) \\ z\in\mathbb{R} \end{matrix}} \vee \boxed{\cancel{\begin{matrix} (0,y,0) \\ y\in\mathbb{R} \end{matrix}}}$$

$$(0,0,z)\in M \; ? \Leftrightarrow \begin{cases} z^2=1 \\ 0=1/4 \end{cases} \text{ imposs}$$

$$(x,y,0)\in M \Leftrightarrow \begin{cases} x^2+y^2=1 \\ x^2+y^2=1/4 \end{cases} \text{ imposs.}$$

$$\Rightarrow \nexists\,(x,y,z)\in M : \operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 2$$

$$\Updownarrow$$

$$\forall\,(x,y,z)\in M \quad \operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2.$$

∎

Thm: (general Lagrange thm)

Let $\vec{x}_0$ be a min/max for f on

$D = \{ g_1(\vec{x})=0,\ g_2(\vec{x})=0, \dots, g_m(\vec{x})=0 \}$.

If $(g_1 - g_m)$ is a submersion at $\vec{x}_0$

$$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \dots + \lambda_m \nabla g_m.$$

Rmk:

$$\nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_m \nabla g_m$$

$$\iff \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ | \\ \nabla g_m \end{bmatrix} \begin{aligned} &= m \\ &< m+1 \end{aligned}$$

$\iff$ all $(m+1) \times (m+1)$ submatrices of

$$\begin{bmatrix} \nabla f \\ \nabla g_1 \\ | \\ \nabla g_m \end{bmatrix} \quad \text{have} \quad \det = 0.$$

Ex. 2.6.17

$$M = \{ (x,y,z) \in \mathbb{R}^3 : \underline{z^2 = x^2+y^2+1}, \ z = \underline{2x^2+y^2} \}$$

Q1 $M \neq \emptyset$ is the zero set of a submersion

Q2 Is M compact?

Q3 Find pts of M at min/max quote (if any)

Sol: Q1 $\quad x = 0 \quad (0,y,z) \in M \iff \begin{cases} z^2 = y^2+1 \\ z = y^2 \end{cases}$

$$z^2 = z+1 \iff z^2 - z - 1 = 0$$

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot (-1)$$

$$>0$$

$$z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$y^2 = z = \frac{1 \pm \sqrt{5}}{2} \lessgtr \begin{matrix} \boxed{\frac{1+\sqrt{5}}{2}} > 0 \\ \frac{1-\sqrt{5}}{2} < 0 \end{matrix}$$

$$y = \pm \sqrt{\frac{1+\sqrt{5}}{2}}$$

$$\left(0, \pm \sqrt{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2}\right). \swarrow$$

$\mathcal{M} = \{g_1 = 0, g_2 = 0\}$ where

$$g_1 = x^2 + y^2 - z^2 + 1$$

$$g_2 = 2x^2 + y^2 - z$$

(g_1, g_2) is a submersion on $\mathcal{M}$ $\Leftrightarrow$

$\nabla g_1, \nabla g_2$ are lin. indep on $\mathcal{M}$ $\Leftrightarrow$

$\operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2$ on $\mathcal{M}$.

Now

$$2 > \operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & -2z \\ 4x & 2y & -1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \det \begin{bmatrix} 2x & 2y \\ 4x & 2y \end{bmatrix} = 0 \\ \det \begin{bmatrix} 2x & -2z \\ 4x & -1 \end{bmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} -\cancel{4}xy = 0 \\ -\cancel{2}x + \overset{4}{\cancel{8}}xz = 0 \\ -\cancel{2}y + \overset{2}{\cancel{4}}yz = 0 \end{cases}$$

$$\det\begin{bmatrix} 2y & -2z \\ 2y & -1 \end{bmatrix} = 0$$

$$\iff \begin{cases} \boxed{xy = 0} \\ x(4z-1) = 0 \\ y(2z-1) = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ \cancel{0 = 0} \\ y(2z-1) = 0 \end{cases} \quad \vee \quad \begin{cases} y = 0 \\ x(4z-1) = 0 \\ \cancel{0 = 0} \end{cases}$$

$$\swarrow$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ z = 1/2 \end{cases} \vee \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = 0 \\ z = 1/4 \end{cases}$$

$(0,0,z)$ $z \in \mathbb{R}$ $\qquad$ $(0,y,1/2)$ $y \in \mathbb{R}$ $\qquad$ $(x,0,1/4)$. $x \in \mathbb{R}$

$$(0,0,z) \in \mathcal{M} \iff \begin{cases} z^2 = 1 \\ z = 0 \end{cases} \text{ imposs.}$$

$$(0,y,1/2) \in \mathcal{M} \iff \begin{cases} \frac{1}{4} = y^2 + 1 \\ \ldots \end{cases} \text{ imposs}$$

$$(x,0,1/4) \in \mathcal{M} \iff \begin{cases} \frac{1}{16} = x^2 + 1 \\ \end{cases} \text{ imposs.}$$

$$\Rightarrow \quad \operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2 \quad \forall (x,y,z) \in \mathcal{M}$$

$\iff$ (g_1, g_2) is a submersion on $\mathcal{M}$.

Q2 $\mathcal{M} = \{ g_1 = 0, g_2 = 0 \}$, $g_1, g_2 \in \mathcal{C} \Rightarrow$

$\mathcal{M}$ is closed.

Is it bounded: $(x,y,z)\in M$

$$\begin{cases} z^2 = \underbrace{x^2+y^2}+1 \\ z = 2x^2+y^2 \end{cases}$$

$$\cancel{\begin{array}{c} x^2+y^2-z^2=-1 \\ \Downarrow \\ x^2\le 1 \\ y^2\le 1 \\ z^2\le 1 \end{array}}$$

$$= x^2+\underbrace{x^2+y^2}_{z^2-1}$$

$$\begin{cases} z = x^2+\boxed{x^2+y^2} \\ x^2+y^2 = z^2-1 \end{cases} \Rightarrow \boxed{x^2+z^2-1=z}$$

$$x^2+z^2-z=1$$

$$(z-\tfrac12)^2\underbrace{-\tfrac14}$$

$$\Rightarrow x^2+(z-\tfrac12)^2-\tfrac14=1$$

$$\Rightarrow x^2+(z-\tfrac12)^2=\tfrac54$$

$$\boxed{x^2\le\frac54}$$

$$-2\le$$

$$-\sqrt{\frac54}\le z-\frac12\le\sqrt{5/4} \Rightarrow \frac12-\sqrt{5/4}\le z\le\frac12+\sqrt{\frac54}\le 2$$

$$\Rightarrow \boxed{z^2\le 4}$$

and from $\cancel{x^2}+y^2 = z^2\underbrace{-1}\underbrace{-x^2}$

$$y^2\le z^2\le 4$$

$$\Rightarrow \quad \|(x,y,z)\|^2\le\frac54+4+4 \quad \forall (x,y,z)\in M.$$

Q3. We have to find $\min/\max_{M} f$

where $f(x,y,z)=z$.

Since M is closed & bded ($\Leftrightarrow$ compact)

$f \in \mathcal{C} \Rightarrow$ (W) $\exists$ both min/max pts.

Let's find these pts: by Q1, we can apply Lagrange thm $\Rightarrow$ at min/max pt on Ⓜ

$$\boxed{\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2}$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} 0 & 0 & 1 \\ 2x & 2y & -2z \\ 4x & 2y & -1 \end{bmatrix} < 3$$

$$\Leftrightarrow \quad \det\begin{bmatrix} 0 & 0 & 1 \\ 2x & 2y & -2z \\ 4x & 2y & -1 \end{bmatrix} = 0$$

$$\Leftrightarrow \quad 1 \cdot \det\begin{bmatrix} 2x & 2y \\ 4x & 2y \end{bmatrix} = 0$$

$$\Leftrightarrow \quad -\cancel{4}\boxed{xy = 0}$$

$$\Leftrightarrow \quad x = 0 \quad \vee \quad y = 0$$

$$\begin{array}{ccc} x = 0 & & y = 0 \\ \Updownarrow & & \\ (0, y, z) & & (x, 0, z) \\ y, z \in \mathbb{R} & & (x, z) \in \mathbb{R}^2 \end{array}$$

$$(0,y,z) \in M \iff \begin{cases} z^2 = y^2 + 1 \\ \boxed{z = y^2} \end{cases}$$

$$\left(0, \pm\sqrt{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2}\right) \xrightarrow{f} \frac{1+\sqrt{5}}{2}$$

max pts.

$$(x,0,z) \in M \iff \begin{cases} z^2 = x^2 + 1 \\ \frac{z}{2} = \cancel{2}x^2 \end{cases}$$

$$z^2 = \frac{z}{2} + 1$$

$$2z^2 - z - 2 = 0$$

$$\Delta = 1 + 16 = 17$$

$$z_{1,2} = \frac{1 \pm \sqrt{17}}{4}$$

$$x^2 = \frac{1 \pm \sqrt{17}}{8} \underset{\times}{=} \frac{1+\sqrt{17}}{8}$$

$$\left(\pm\sqrt{\frac{1+\sqrt{17}}{8}}, 0, \frac{1+\sqrt{17}}{4}\right) \to \frac{1+\sqrt{17}}{4}$$

min pts ■

Do

Ex 2.6.20/21/22/27/26/28

2.6.21

$$M = \{ \overbrace{x^2 - xy + y^2}^{=1} - z^2 = 1, \quad x^2 + y^2 = 1 \}$$

Q1 $M \neq \emptyset$, M = zero set of a subm.

Q2 Is M compact

Q3 max / min f $\boxed{f = xyz}$

M

Q1 $\mathcal{M} = \{ x^2 + y^2 = 1 , + xy + z^2 = 0 \}$

$$(0, y, z) \in \mathcal{M} \iff \begin{cases} y^2 = 1 \\ z^2 = 0 \end{cases}$$

$x = 0$

$$(0, \pm 1, 0) \in \mathcal{M}$$

$\mathcal{M} = \{ g_1 = 0 , g_2 = 0 \}$ $\quad g_1 = x^2 + y^2 - 1$

$g_2 = xy + z^2$

(g_1, g_2) is submersion on $\mathcal{M} \iff$

$\nabla g_1 , \nabla g_2$ are lin. indep on $\mathcal{M} \iff$

$$\operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 2 .$$

Now, $\operatorname{rk} \begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rk} \begin{bmatrix} 2x & 2y & 0 \\ y & x & 2z \end{bmatrix} < 2$

$$\iff \begin{cases} \det \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix} = 0 \\ \det \begin{bmatrix} 2x & 0 \\ y & 2z \end{bmatrix} = 0 \\ \det \begin{bmatrix} 2y & 0 \\ x & 2z \end{bmatrix} = 0 \end{cases} \iff \begin{cases} \cancel{2}x^2 - \cancel{2}y^2 = 0 \\ \cancel{4}xz = 0 \\ \cancel{4}yz = 0 \end{cases}$$

$$\iff \begin{cases} (y - x)(y + x) = 0 \\ xz = 0 \\ \boxed{yz = 0} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 0 \\ x\cancel{z=}0 \\ -x^2=0 \end{cases} \quad 0=0 \qquad \vee \qquad \begin{cases} z = 0 \\ 0 = 0 \\ (y-x)(y+x)=0 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \swarrow$$

$$x = 0 \qquad\qquad \begin{cases} z=0 \\ y = x \end{cases} \quad \vee \quad \begin{cases} z = 0 \\ y = -x \end{cases}$$

$$(0,0,z) \quad z\in\mathbb{R} \qquad\qquad (x,x,0) \quad (x,-x,0)$$

$$x\in\mathbb{R}$$

$$(0,0,z)\in \mathcal{M} \Leftrightarrow \begin{cases} -z^2=1 \\ \end{cases} \qquad \text{imposs}$$

$$(x,x,0)\in\mathcal{M} \Leftrightarrow \begin{cases} x^2+0^2=0 \\ x^2+x^2=1 \end{cases} \quad \begin{matrix} x^2=0 \\ 2x^2=1 \end{matrix}$$

imposs.

$$(x,-x,0)\in\mathcal{M} \Leftrightarrow \begin{cases} -x^2=0 \\ x^2+x^2=1 \end{cases} \qquad \text{imposs.}$$

$\Rightarrow$ on $\mathcal{M}$, (g_1,g_2) is a submersion

Q2. $\mathcal{M} = \{g_1=0,\ g_2=0\}$, $g_1,g_2\in\mathcal{C}$

$\Rightarrow$ $\mathcal{M}$ is closed

Bounded?

$$(x,y,z)\in\mathcal{M} \Leftrightarrow \begin{cases} xy+z^2=0 \leftarrow \\ \boxed{x^2+y^2=1} \end{cases}$$

$$\downarrow$$

$$\boxed{x^2\le 1},\ \boxed{y^2\le 1}$$

$$z^2 = -xy \leq 1$$

$$\Rightarrow \boxed{z^2 \leq 1}$$

$$\Rightarrow \|(x,y,z)\| \leq \sqrt{3} \quad \forall (x,y,z) \in \mathcal{M}.$$

Q3. $\mathcal{M}$ is compact (by Q2), $f \in \mathcal{C}$

$\Rightarrow$ (W) $\exists \min/\max_{\mathcal{M}} f$.

Let's find them: by Q1 we can apply Lagrange thm: at min/max pts for f on $\mathcal{M}$ we have

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3$$

$$\Leftrightarrow \quad \det\begin{bmatrix} \begin{matrix} yz & xz \\ 2x & 2y \\ y & x \end{matrix} & \boxed{\begin{matrix} xy \\ 0 \\ 2z \end{matrix}} \end{bmatrix} = 0$$

$$0 = xy \det\begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix} + 2z \det\begin{bmatrix} yz & xz \\ 2x & 2y \end{bmatrix}$$

$$0 = xy\,\cancel{2}(x^2-y^2) + \cancel{2}z^2\, 2(y^2-x^2)$$

$$0 = (y^2-x^2)(2z^2-xy)$$

$$0 = (y-x)(y+x)(2z^2-xy)$$

$\Updownarrow$

$y = x \qquad (x, x, z) \in M \iff \begin{cases} 2x^2 = 1 \\ x^2 + z^2 = 0 \end{cases}$ impos

$\vee$

$y = -x \qquad (x, -x, z) \in M \iff \begin{cases} 2x^2 = 1 \\ -x^2 + z^2 = 0 \end{cases} \quad z^2 = x^2 \Rightarrow z = \pm x$

$x = \pm \frac{1}{\sqrt{2}}$

$\vee$

$\underline{2z^2 - xy = 0 \; : \; (x, y, z) \in M}$

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) \quad \begin{matrix} + - - \\ + - + \end{matrix}$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) \quad \begin{matrix} - + - \\ - + + \end{matrix}$

max

min

$$\begin{cases} 2z^2 - xy = 0 \\ x^2 + y^2 = 1 \\ xy + z^2 = 0 \end{cases} \qquad \begin{cases} xy = 2z^2 = 0 \\ x^2 + y^2 = 1 \\ \cancel{3}z^2 = 0 \end{cases}$$

$\Updownarrow$

$$\begin{cases} x = 0 \\ y = \pm 1 \\ z = 0 \end{cases} \quad \vee \quad \begin{cases} y = 0 \\ x = \pm 1 \\ z = 0 \end{cases}$$

$$(0, \pm 1, 0) \xrightarrow{f} 0 \xleftarrow{f} (\pm 1, 0, 0)$$