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Important Rmk

If f is int on D $\left(\exists \int_D f\right)$

then RF applies

$$\Rightarrow \int_D f(x,y)\,dx\,dy = \int_{\mathbb{R}} \int_{D^y} f(x,y)\,dx\,dy$$

$D_x \qquad D^y_{\cancel{x}} = \{x : (x,y)\in D\}$

$\qquad \overset{\shortparallel}{\{y : (x,y)\in D\}}$

$$= \int_{\mathbb{R}} \int_{D_x} f(x,y)\,dy\,dx$$

However, it can happen that

$$\exists \int_{\mathbb{R}} \int_{D^y} f\,dx\,dy, \quad \exists \int_{\mathbb{R}} \int_{D_x} f\,dy\,dx$$

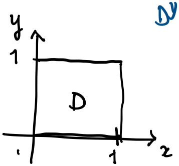
and they are different (in this case, necessarily, $\nexists \int_D f$ and $\int_D |f| = +\infty$)

Ex 4.3.4

Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$ $\quad (x,y)\in D = [0,1]^2$

Let's compute $\displaystyle \int\left(\int_{D^y} f\,dx\right)dy, \ \int\left(\int_{D_x} f\,dy\right)dx$

Sol:



$$D = \left\{(x,y) ; \boxed{0\le x\le 1} \atop \boxed{0\le y\le 1}\right\}$$

$$\int_0^1\left(\int_0^1 \frac{\overset{x+y-y}{x-y}}{(x+y)^3}\,dx\right)dy = \int_0^1 \boxed{\left(\int_0^1 \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\,dx\right)}dy \overset{(*)}{=}$$

$$\int_0^1 \frac{1}{(x+y)^2}\,dx = \int_0^1 (x+y)^{-2}\,dx$$

$$\parallel$$

$$\partial_x\left[-(x+y)^{-1}\right]$$

$$= \left[\ominus (x+y)^{-1}\right]_{x=0}^{x=1} = \boxed{-\left(\frac{1}{y+1}-\frac{1}{y}\right)}$$

$$\int_0^1 \frac{-2y)}{(x+y)^3}\,dx = -2y\int_0^1 (x+y)^{-3}\,dx$$

$$\partial_x \frac{(x+y)^{-2}}{-2} = \cancel{-2}(x+y)^{-3}$$

$$= \cancel{-2}y\left[\frac{(x+y)^{-2}}{\cancel{-2}}\right]_{x=0}^{x=1}$$

$$= y\left[\frac{1}{(y+1)^2}-\frac{1}{y^2}\right] = \boxed{\frac{y}{(y+1)^2}-\frac{1}{y}}$$

$$\overset{(*)}{=} \int_0^1 \left(\cancel{\frac{1}{y}} - \frac{1}{y+1} + \frac{y+1-1}{(y+1)^2} - \cancel{\frac{1}{y}}\right)dy$$

$$= \int_0^1 -\cancel{\frac{1}{y+1}} + \cancel{\frac{1}{y+1}} - \frac{1}{(y+1)^2}\,dy$$

$$= +\int_0^1 \frac{1}{(y+1)^2}\,dy$$

$$\parallel$$

$$-(y+1)^{-2} = \partial_y (y+1)^{-1}$$

$$= \left[(y+1)^{-1}\right]_{y=0}^{y=1} = \frac{1}{2} - 1 = -1/2.$$

$$\Rightarrow \quad \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dx\, dy = -\tfrac{1}{2}.$$

Now

$$\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dy\, dx$$

$$= \int_0^1 \int_0^1 \frac{y-x}{(y+x)^3}\, dx\, dy$$

$$= +\tfrac{1}{2} \qquad !$$

■

General R.F.

What if we have to compute an integral

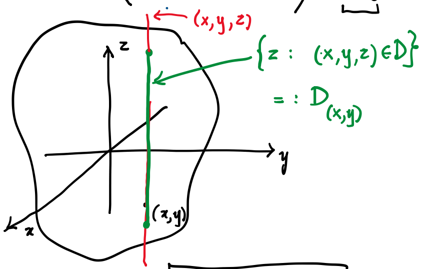
$$\int_D f(x,y,z)\, dx dy dz$$

Now the idea behind RF for two vars is

$$\int_D f(x,y)\, dx\, dy = \int_{\mathbb{R}} \int_{D_x} f(x,y)\, \overset{\text{first int var}}{dy}\, \underset{\text{second int var}}{dx}$$

For $\int_D f(x,y,z)\cdot dx\, dy\, dz$ we may say

$$\int \Big(\overbrace{\int f(x,y,z)\, dz}^{\text{funct of } (x,y)} \Big)\, \underbrace{dx\, dy}_{}$$



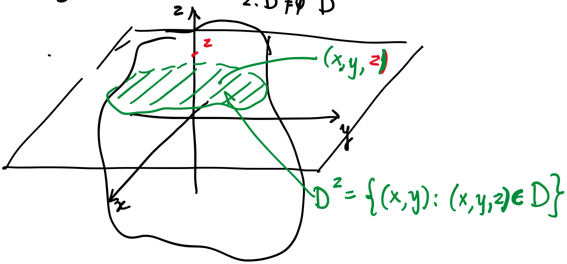
$\{z : (x,y,z) \in D\}$

$=: D_{(x,y)}$

$$\int_D f\, dx dy dz = \int_{(x,y) : D_{(x,y)} \neq \emptyset} \boxed{\int_{D_{(x,y)}} f(x,y,z)\, dz}\, dx dy$$

funct of (x,y)

$$\int_D f(x,y,z)\,dxdydz = \int_{z:\,D^z\neq\emptyset}\left(\int_{D^z} f(x,y,z)\,dxdy\right)dz$$

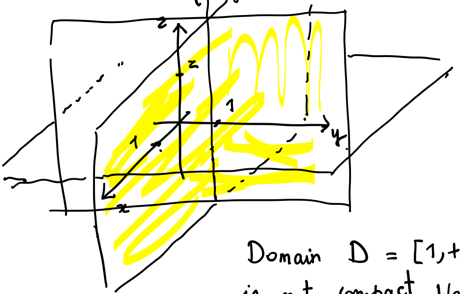


Ex 4.7.2.

#1 $\displaystyle\int_{[1,+\infty[^3} y^3 z^8 e^{-xy^2z^3}\,dxdydz$

$$[1,+\infty[^3 = [1,+\infty[\times[1,+\infty[\times[1,+\infty[$$

$$= \{(x,y,z) : x\geqslant 1,\ y\geqslant 1,\ z\geqslant 1\}$$



Domain $D = [1,+\infty[^3$ is not compact. We should check integrability. Since $|f| = f \geqslant 0$, we need just to check one of the iterated integrals of f is finite

$$\int_D f = \int_{\substack{y\geq 1\\ z\geq 1}} \left(\int_1^{+\infty} y^3 z^8 e^{-xy^2z^3} dx \right) dy\, dz$$

$$= \int_{\substack{y\geq 1\\ z\geq 1}} y^3 z^8 \boxed{\int_1^{+\infty} e^{-xy^2z^3} dx} \, dy\, dz \overset{(*)}{=}$$

$$\partial_x \frac{e^{-xy^2z^3}}{-y^2z^3} = e^{-xy^2z^3} (\cancel{-y^2z^3})$$

$$\int_1^{+\infty} e^{-xy^2z^3} dx = \left[\frac{e^{-xy^2z^3}}{\underbrace{-y^2z^3}} \right]_{x=1}^{x=+\infty}$$

$$= -\frac{1}{y^2z^3}\left(0 - e^{-y^2z^3}\right)$$

$$= \frac{1}{y^2z^3} e^{-y^2z^3}$$

$$\overset{(*)}{=} \int_{y\geq 1, z\geq 1} y^{\cancel{3}} z^{\cancel{8}\,5} \cdot \frac{1}{\cancel{y^2}\cancel{z^3}} e^{-y^2z^3} dy\, dz$$

$$= \int_{y\geq 1,\, z\geq 1} y z^5 e^{-y^2z^3} dy\, dz$$

$$\overset{RF}{=} \int_1^{+\infty} \left(\int_1^{+\infty} y z^5 e^{-y^2z^3} dy \right) dz$$

$$= \int_1^{+\infty} z^5 \boxed{\int_1^{+\infty} y e^{-y^2z^3} dy} \, dz \overset{(**)}{=}$$

$$\partial_y \frac{e^{-y^2z^3}}{-2z^3} = e^{-y^2z^3} \cdot (\cancel{-2}y\cancel{z^3})$$

$$\int_1^{+\infty} y e^{-y^2z^3} dy = \left[\frac{e^{-y^2z^3}}{\underbrace{-2z^3}} \right]_{y=1}^{y=+\infty}$$

$$= -\frac{1}{2z^3}\left(0 - e^{-z^3}\right) = \frac{1}{2z^3} e^{-z^3}$$

$$\overset{(**)}{=} \int_1^{+\infty} z^{\cancel{5}2} \frac{1}{2\cancel{z^3}} e^{-z^3} dz$$

$$= \frac{1}{2} \int_1^{+\infty} z^2 e^{-z^3} dz$$

$$\partial_z \frac{e^{-z^3}}{-3} = e^{-z^3} \cdot (\cancel{-3}z^2)$$

$$= \frac{1}{2} \left[\frac{e^{-z^3}}{-3} \right]_{z=1}^{z=+\infty} = -\frac{1}{6}\left(0 - e^{-1}\right)$$

$$= \frac{1}{6e}.$$

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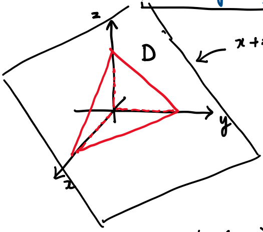
Do 4.7.2.

# 2 $$\int_{x \geq 0,\ y \geq 0,\ \boxed{z \geq 0},\ \boxed{x+y+z \leq 1}} \boxed{xyz}\ dx\,dy\,dz$$

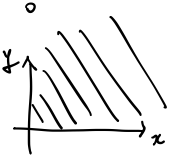


$x+y+z=1$

$z \leq 1-(x+y)$

$\boxed{0 \leq z \leq 1-(x+y)}$

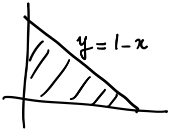
$$= \int_{\substack{x \geq 0 \\ y \geq 0 \\ 1-(x+y) \geq 0}} \int_0^{1-(x+y)} f \quad dz \quad dx\,dy$$



$1-(x+y) \geq 0$

$x+y \leq 1$

$y \leq 1-x$



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Change of vars.

Recall that

$$\int_a^b \underbrace{f(x)}\,dx \underset{y=\phi(x)}{=} \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy$$

$$x = \phi^{-1}(y) \leftarrow$$

$$dx = (\phi^{-1})'(y)\,dy$$

$$x = a$$

$$\downarrow$$

$$y = \phi(x) \overset{x=a}{=} \phi(a)$$

$$\int f(x)\,dx \underset{y=\phi(x)}{=} \int f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy$$

Ex:

$$\int_0^4 e^{\sqrt{x}}\,dx = \int_0^2 e^{y}\,2y\,dy$$

$$y = \sqrt{x} = \phi(x)$$

$$x = \underbrace{y^2} = \phi^{-1}(y)$$

$$dx = 2y\,dy$$

$$= \ldots$$

$$\lrcorner$$

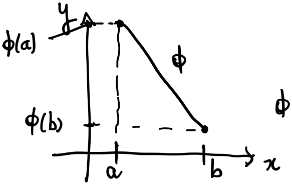
Now the change of var $y = \phi(x)$ can be only of two types:

- $\phi \nearrow$ (increasing)

$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))\underset{+}{\left|(\phi^{-1})'(y)\right|}\,dy$$

- $\phi \searrow$ (decreasing)

$$\int_a^b f(x)\,dx = \int_{\boxed{\phi(a)}}^{\boxed{\phi(b)}} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy$$



$y = \phi(x)$ ↘

$\phi(b) < \phi(a)$ ⇓

ϕ^{-1} ↘

$$= \ominus \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y)) \boxed{(\phi^{-1})'(y)} dy$$

$\ominus\ominus$

$$= \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y)) \left|(\phi^{-1})'(y)\right| dy$$

This leads to a unified formula

$$\boxed{\int_{[a,b]} f(x)\,dx = \int_{\phi([a,b])} f(\phi^{-1}(y))\left|(\phi^{-1})'(y)\right|dy}$$

$y = \phi(x)$

$x = \phi^{-1}(y)$

Thm: $f = f(\vec{x})$ $\vec{x} \in \mathbb{R}^n$

$$\int_D f(\vec{x})\,d\vec{x} \underset{\substack{\vec{y} = \Phi(\vec{x}) \\ \vec{x} = \Phi^{-1}(\vec{y})}}{=} \int_{\Phi(D)} f(\phi^{-1}(\vec{y}))\left|\det(\phi^{-1})'(\vec{y})\right| d\vec{y}$$

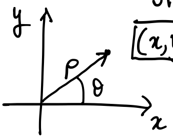
$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$\begin{tikzcd}
\vec{x} \in \mathbb{R}^n \arrow[r, bend left] & \vec{y} \in \mathbb{R}^n \arrow[l, bend left]
\end{tikzcd}$$

$\phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

We now apply this formula to $\int_D f(x,y)\,dx\,dy$

Integration in Polar coords



old new

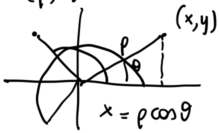
$$\boxed{(x,y)} \rightleftarrows \boxed{(\rho,\vartheta)}$$

$\rho \geq 0, \quad \theta \in [0, 2\pi[$

$$\boxed{\phi^{-1}\begin{cases} x = \boxed{\rho\cos\vartheta} \leftarrow \\ y = \rho\sin\theta \leftarrow \end{cases}}$$

old funct of new

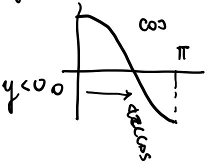
$$(x,y) = \phi^{-1}(\rho,\vartheta)$$

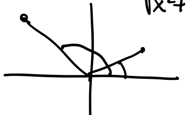


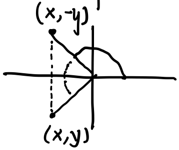
$\Downarrow$

$$\cos\theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\phi : \begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \begin{cases} \arccos \boxed{\dfrac{x}{\sqrt{x^2+y^2}}} & y \geq 0 \\ \dfrac{\pi}{2} + \arccos \dfrac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \end{cases}$$







Fortunately we don't need ϕ, but ϕ^{-1}

$$\det(\phi^{-1})' = \begin{array}{c} \begin{array}{cc} \partial_\rho & \partial_\theta \end{array} \\ \begin{bmatrix} \nabla \phi_1^{-1} \\ \nabla \phi_2^{-1} \end{bmatrix} \end{array} \overset{\det}{=} \begin{bmatrix} \cos\theta & \ominus\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix}$$

$$\phi_1^{-1} = \rho\cos\vartheta$$

$$\phi_2^{-1} = \rho\sin\theta$$

$$= \rho\cos^2\theta + \rho\sin^2\theta$$

$$= \rho$$

$$\Rightarrow \quad \boxed{\left|\det\left((\phi^{-1})'\right)\right| = \rho}$$

So

$$\int_D f(x,y)\,dx\,dy = \int_{\underbrace{\phi(D)}_{\text{D written in polar coords}}} f(\rho\cos\theta,\rho\sin\theta)\,\rho\,d\rho\,d\theta$$

Ex 4.4.3. Compute

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\,dx\,dy$$

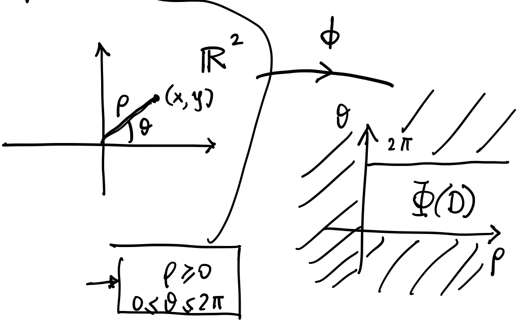
Sol: $D = \mathbb{R}^2$ not compact, $f \geqslant 0$

$$\int_{\mathbb{R}^2} \underbrace{e^{-\sqrt{x^2+y^2}}}\,dx\,dy \overset{RF}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\sqrt{x^2+y^2}}\,dx\,dy$$

?

$$\underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \int_{?} e^{-\rho}\,\rho\,d\rho\,d\theta$$

$\sqrt{x^2+y^2} = \rho$



$$= \int_{\rho\geqslant 0,\ \boxed{0\leqslant\theta\leqslant 2\pi}} \rho\, e^{-\rho}\,d\rho\,d\theta$$

$$\overset{RF}{=} \int_0^{+\infty}\int_0^{2\pi} \rho e^{-\rho}\, d\theta\, d\rho$$

$$= \int_0^{+\infty} \rho e^{-\rho} \underset{\substack{\parallel \\ 2\pi}}{\int_0^{2\pi} 1\, d\theta}\, d\rho$$

$$= 2\pi \int_0^{+\infty} \underset{\substack{\parallel \\ \partial_\rho(-e^{-\rho})}}{\rho e^{-\rho}}\, d\rho = 2\pi \left[\overset{\substack{0 \\ /\!/}}{-\rho e^{-\rho}} \Big|_{\rho=0}^{\rho=+\infty} + \int_0^{+\infty} 1\cdot(+e^{-\rho})\, d\rho \right]$$

$$= 2\pi \int_0^{+\infty} \underset{\substack{\parallel \\ \partial_\rho - e^{-\rho}}}{e^{-\rho}}\, d\rho = -2\pi \underset{0-1}{\left[\cancel{\rho} e^{-\rho} \right]_0^{+\infty}}$$

$$= -2\pi(-1) = 2\pi \quad \blacksquare$$

Ex. (Gaussian Integral)

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

Sol: Call

$$\to I = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx > 0$$

$$I^2 = I \cdot I = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy$$

$$= \int_{-\infty}^{+\infty} \left(e^{-\frac{x^2}{2}} \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy \right) dx$$

$$= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\, dy \right) dx$$

$$\boxed{I^2} \overset{RF}{=} \int_{\underline{\mathbb{R}^2}} e^{-\frac{\boxed{x^2+y^2}}{2}} \, dx\,dy$$

$$|\det(\phi^{-1})'|$$

$$\underset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{=} \int_{\substack{\rho \geq 0 \\ 0 \leq \theta \leq 2\pi}} e^{-\frac{\rho^2}{2}} \rho \, d\rho\, d\theta$$

$$\overset{RF}{=} \int_0^{2\pi} \underbrace{\left(\int_0^{+\infty} \rho\, e^{-\rho^2/2} \, d\rho \right)}_{} d\theta$$

$$\partial_\rho \frac{e^{-\rho^2/2}}{-1} = e^{-\rho^2/2} \cdot (\cancel{-}\rho)$$

$$= \int_0^{2\pi} -\left[+ e^{-\rho^2/2} \right]_{\rho=0}^{\rho=+\infty} d\theta$$

$$0 - 1$$

$$= \int_0^{2\pi} 1 \, d\theta = \boxed{2\pi.}$$

$$\Rightarrow I^2 = 2\pi \Rightarrow I = \cancel{\pm}\sqrt{2\pi} \quad \blacksquare$$

Do Ex 4.7.3 (compute the int.)

Ex 4.7.5 #1,2